CLT of Wigner Ensemble: A Combinatorial Way

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1 Motivation

The seminal result at the beginning of Random Matrix Theory is the *Wigner semicircle law*, which states: For the Wigner matrix ensemble (scaled by $1/\sqrt{n}$), the empirical spectral distribution (ESD):

$$\nu_n = \frac{1}{n}\sum_{i=1}^{n} \delta_{\lambda_i},$$

which is a random probability measure on $\mathbb{R}$, converges weakly, almost surely to the semicircle distribution μ_{sc}:

$$\mu_{\text{sc}}(dx) = \frac{1}{2\pi}\sqrt{4-x^2}\,dx, \qquad x \in [-2,2].$$

This is sometimes viewed as the *law of large numbers* for Wigner matrices; the average of random measures goes to their mean, which is deterministic. More precisely, one can consider a.s. convergence of random variables:

$$\int_{\mathbb{R}} f(x)\,\nu_n(dx), \quad f \in C_b(\mathbb{R}).$$

That will give the *law of large numbers* for linear statistics[1].

As a routine for probabilists, we are led to consider its fluctuation and, further, the **central limit theorem (CLT)**. However, it seems impossible to discuss these for random measures.

[1] Let $\lambda_i' s$ be the eigenvalues of a random matrix. The linear statistic associated with a test function f is defined as: $L_N(f) = \sum_{i=1}^{N} f(\lambda_i)$. The function f can be chosen to extract various aspects of the spectrum, for instance, $f = x^p$ for the p-th moments of the ESD, while more complex functions can be used to probe fluctuations or localized spectral behavior.

One way to move on is to consider their linear statistics, especially their moments. Let's first consider the latter one:

$$m_p^{(n)} = \int_{\mathbb{R}} x^p \, \nu_n(dx) = \frac{1}{n} \sum \lambda_i^p = \frac{1}{n} \operatorname{Tr}(A^p).$$

In the combinatorial proof of the semicircle law, we have shown by the super concentration $\operatorname{Var}(\frac{1}{n} \operatorname{Tr} A^p) = O(n^{-2})$ that a.s.

$$\frac{1}{n} \operatorname{Tr} A^p \to \mu_p = \begin{cases} \frac{(2s)!}{s!(s+1)!}, & \text{if } p = 2s, \\ 0, & \text{if } p = 2s+1. \end{cases} \tag{1.1}$$

and due to strong correlations between eigenvalues[2], the fluctuation is of order $\frac{1}{n}$. So the right thing to analyze here is

$$\operatorname{Tr} A^p - \mathbb{E} \operatorname{Tr} A^p.$$

Lots of information about fluctuation is neglected before scaled by n. If one counts more carefully, by a large extent Jonsson's method[Jon82],[AGZ10, Section 2.1.7], the CLT can be proved that $\operatorname{Tr} A^p - \mathbb{E} \operatorname{Tr} A^p$ converge in distribution to $\mathcal{N}(0, \sigma_p)$ for any fixed p. A by-product of this is the CLT for linear statistics associated with polynomials.

Under some mild assumptions, [SS98] gives a much fancier version of CLT, where the power p is growing at a slower rate than $\sqrt{n}$, by the so-called *Sinai-Soshniko technique*. With this in hand, one can show the CLT for linear statistics associated with holomorphic functions. This is an improvement of the work by Z.Füredi and J.Komlós[FK81], who discussed the case $p = o(n^{-\frac{1}{6}})$. Our main task in this article is to present the Sinai-Soshniko technique in detail.

2 Main Results

Consider **Real Wigner matrices**, where the components $a_{ij} = a_{ji} = \frac{\xi_{ij}}{\sqrt{n}}$ of the symmetric $n \times n$ matrices A are such that:

1. $\{\xi_{ij}\}_{1 \le i \le j \le n}$ are independent random variables;
2. The laws of distribution for ξ_{ij} are symmetric, hence all odd moments of ξ_{ij} vanish;
3. Each moment $\mathbb{E}\xi_{ij}^p$ exists and $\mathbb{E}|\xi_{ij}^p| \le C_p$, C_p is a constant depending only on p;
4. The second moments of ξ_{ij}, $i < j$, are equal $\frac{1}{4}$; for $i = j$ they are uniformly bounded[3].

We will prove:

Theorem 1 (Main Theorem)**.** *Consider real symmetric Wigner ensemble with the additional assumption*

$$\mathbb{E}\, \xi_{ij}^{2k} \le (const\, k)^k, \qquad const > 0 \tag{2.1}$$

uniformly in i, j and k, meaning that the moments of ξ_{ij} grow not faster than the Gaussian. Then

$$\mathbb{E}(\operatorname{Tr} A^p) = \begin{cases} \frac{1}{\sqrt{\pi}} \frac{n}{s^{3/2}} (1 + o(1)), & p = 2s \\ 0, & p = 2s+1 \end{cases} \tag{2.2}$$

[2]One can consider the vandermonde term in $G\beta E$.

[3]So that the limiting spectrum will be modified to unit interval.

and $\mathrm{Tr}\, A^p - \mathbb{E}(\mathrm{Tr}\, A^p)$ *converges in distribution to* $\mathcal{N}(0, \frac{1}{\pi})$*, as long as* $1 \ll p \ll \sqrt{n}$*.*

Moreover, define $\lfloor e^t p \rfloor$ *as the nearest integer* p' *to* $e^t p$ *such that* $p' - p$ *is even, then the random process*

$$\eta_p(t) = \mathrm{Tr}\, A^{\lfloor e^t p \rfloor} - \mathbb{E}\, \mathrm{Tr}\, A^{\lfloor e^t p \rfloor}$$

converges in the finite-dimensional distributions to the stationary (Gaussian) process $\eta(t)$ *with zero mean and covariance function:*

$$\mathrm{Cov}(\eta(t_1), \eta(t_2)) = \mathbb{E}\, \eta(t_1) \cdot \eta(t_2) = \frac{1}{\pi \cosh\left(\frac{1}{2}(t_1 - t_2)\right)}. \tag{2.3}$$

Remark. For fixed s, by the asymptotic of Catalan number $C_s \sim \frac{4^s}{s^{3/2}\sqrt{\pi}}$, (2.2) is consistent with (1.1). However, the weak limit for fixed p differs that for growing p, it converges in distribution to $\mathcal{N}(0, \sigma_P)$, where σ_p is determined by p, see [AGZ10]. There is also finite-dimensional limit for fixed exponents [AZ06].

Remark. The finite-dimensional limit here doesn't depend on the fourth and higher moments of ξ_{ij} and rate of growth of p. This supports the conjecture of the local universality of the distribution of eigenvalues in different ensembles of random matrices.

Remark. The technique used in [SS98] can be modified to extend the results to the case of not necessarily symmetrically distributed random entries. The main result can also be extended to the **complex Wigner ensemble** and **Wishart matrices**, which means it's valid for *GOE* and *GUE*.

3 Applications

Before digging further into the combinatorics, the first application here concerns the rate of convergence of the maximal eigenvalue. Z. D. Bai and Y. Q. Yin showed in [BY88] the a.s. convergence of $\lambda_{\max}$ to 1 assuming only the existence of the fourth moments of ξ_{ij}, where the main ingredient is the estimate of the mathematical expectations of traces of high powers of A. In [TW96], C. Tracy and H. Widom proved that for Gaussian Orthogonal Ensemble[4]

$$\lambda_{\max}(A) = 1 + O(n^{-2/3})$$

and calculated the famous *GOE Tracy-Widom* distribution

$$G(x) = \lim_{n\to\infty} \mathbb{P}\left\{\lambda_{\max} < 1 + \frac{x}{n^{2/3}}\right\}.$$

With our main theorem, we have:

Corollary 2. *Under the conditions of the Main Theorem*

$$\lambda_{\max}(A) = 1 + o(n^{-1/2} \log^{1+\epsilon} n)$$

for any $\epsilon > 0$ *and with probability 1.*

[4]Here we have pre-scaled the matrix by $\frac{1}{2\sqrt{n}}$.

Proof. Choose

$$p = 2\left[\frac{1}{2}\frac{n^{1/2}}{\log^{\epsilon/2} n}\right], \quad \forall \epsilon > 0.$$

Then

$$\begin{aligned}
\mathbb{P}\left\{\lambda_{\max}(A) \geq 1 + \frac{\log^{1+\epsilon} n}{n^{1/2}}\right\} &\leq \mathbb{P}\left\{\operatorname{Tr}(A^p) \geq \left(1 + \frac{\log^{1+\epsilon} n}{n^{1/2}}\right)^p\right\} \\
&\leq \mathbb{P}\left\{\operatorname{Tr}(A^p) \geq \frac{1}{2}\exp\left(\log^{1+\epsilon/2} n\right)\right\} \\
&\leq \frac{\mathbb{E}\operatorname{Tr} A^p}{\frac{1}{2}\exp\left(\log^{1+\epsilon/2} n\right)} \\
&= o\left(n \exp\left(-\log^{1+\epsilon/2} n\right)\right),
\end{aligned}$$

where the second inequality is by $\log(1+x) \geq x - \frac{x^2}{2}$ for $x > 0$ and it's at least valid for large n. It implies

$$\sum_{n=1}^{\infty} \mathbb{P}\left\{\lambda_{\max}(A) \geq 1 + \frac{\log^{1+\epsilon} n}{n^{1/2}}\right\} < \infty.$$

From Borel-Cantelli lemma,

$$\lambda_{\max}(A) = 1 + O(n^{-1/2}\log^{1+\epsilon} n) \quad a.s.$$

hence $\lambda_{\max}(A) = 1 + o(n^{-1/2}\log^{1+\epsilon} n)$ since ϵ is arbitrary.

□

Remark. The interesting fact here is that one uses a result of global statistics, which is in the Gaussian universality class, to prove a result for local statistics, which is central in the KPZ universality class.

As a crucial application, we now emphasize that:

Corollary 3. *Let $f(z)$ be an analytic function on a neighborhood of the closed unit disk $|z| \leq 1$. Then* [5]

$$\sum_{i=1}^{n} f(\lambda_i) - \mathbb{E}\left(\sum_{i=1}^{n} f(\lambda_i)\right)$$

converges in distribution to the Gaussian $\mathcal{N}(0, \sigma_f)$.

Proof. Denote the linear statistics by

$$S_n(f) := \sum_{i=1}^{n} f(\lambda_i),$$

and

$$E = \left\{|\lambda_i| \leq 1 + \frac{\log^{1+\epsilon} n}{\sqrt{n}}; i = 1, \ldots, n\right\}.$$

[5] A more precise way is to take value 0 for λ out of the domain.

We first truncate it, write:

$$S_n(f) = S_n(f)\mathbf{1}_E + S_n(f)\mathbf{1}_{E^c}.$$

It follows from Corollary 1 that probability of E^c decays faster than any power of n and since $S_n(f) = O(n)$,

$$\mathbb{E}S_n(f) - \mathbb{E}\left(S_n(f)\mathbf{1}_E\right) \underset{n\to\infty}{\longrightarrow} 0.$$

It suffices to prove the Central Limit Theorem for $S_n(f)\mathbf{1}_E$. Write the Taylor series for $f(x)$:

$$f(x) = \sum_{k=0}^{\infty} a_k x^k,$$

and hence

$$S_n(f)\mathbf{1}_E = \sum_{k=0}^{\infty} a_k \operatorname{Tr} A^k \mathbf{1}_E$$

by absolute convergence in the convergent circle. An immediate idea is to use our result for traces.

Fix a large enough M and write:

$$\begin{aligned} S_n(f)\mathbf{1}_E - \mathbb{E}(S_n(f)\mathbf{1}_E) &= \sum_{k=0}^{\infty} a_k \left(\operatorname{Tr} A^k \mathbf{1}_E - \mathbb{E}\left(\operatorname{Tr} A^k \mathbf{1}_E\right)\right) \\ &= \sum_{k=0}^{M} a_k \left(\operatorname{Tr} A^k \mathbf{1}_E - \mathbb{E}\left(\operatorname{Tr} A^k \mathbf{1}_E\right)\right) \\ &+ \sum_{k=M+1}^{n^{1/10}} a_k \left(\operatorname{Tr} A^k \mathbf{1}_E - \mathbb{E}\left(\operatorname{Tr} A^k \mathbf{1}_E\right)\right) \\ &+ \sum_{k>n^{1/10}} a_k \left(\operatorname{Tr} A^k \mathbf{1}_E - \mathbb{E}\left(\operatorname{Tr} A^k \mathbf{1}_E\right)\right). \end{aligned}$$

Cauchy's estimate implies the exponential decay of the series coefficients:

$$|a_k| \le c(1-\delta)^k, \quad c = c(\delta) > 0, \quad 0 < \delta < 1.$$

It will be proved later that

$$\operatorname{Var}\left(\operatorname{Tr} A^p\right), \quad p \ll \sqrt{n}$$

are uniformly bounded. We first do some estimates for the variance. Let $b_n^k = \operatorname{Tr} A^k \mathbf{1}_E - \mathbb{E}\left(\operatorname{Tr} A^k \mathbf{1}_E\right)$,

$$\operatorname{Var}\left(\sum_{k=0}^{M} a_k \left(\operatorname{Tr} A^k \mathbf{1}_E - \mathbb{E}\left(\operatorname{Tr} A^k \mathbf{1}_E\right)\right)\right) = \sum_{k,l=0}^{M} a_k a_l \operatorname{Cov}(b_n^k, b_n^l),$$

$$|RHS| \le \sum_{k,l=0}^{M} |a_k a_l| \sqrt{\operatorname{Var}(b_n^k)\operatorname{Var}(b_n^l)} = \left(\sum_{k=0}^{M} |a_k| \sqrt{\operatorname{Var}(b_n^k)}\right)^2 \le c'(\sum_{k=0}^{M}(1-\delta)^k)^2.$$

Together with the exponential decay, it yields that the variance of the first subsum is bounded. This statement is also valid if we substitute M with $n^{1/10}$, hence (by DCT)

$$\lim_{n\to\infty} \mathrm{Var}\left(\sum_{k=0}^{n^{1/10}} a_k \left(\mathrm{Tr}\, A^k \mathbf{1}_E - \mathbb{E}\left(\mathrm{Tr}\, A^k \mathbf{1}_E\right)\right)\right) = \sum_{k,l=0}^{\infty} a_k a_l \lim_{n\to\infty} \mathrm{Cov}(b_n^k, b_n^l) < \infty,$$

where the existence of limits in RHS is from the main theorem, and we denote this whole limit σ_f. Similarly, we have variance for the second subsum[6]

$$\mathrm{Var}\left(\sum_{k=M+1}^{n^{1/10}} a_k \left(\mathrm{Tr}\, A^k \mathbf{1}_E - \mathbb{E}\left(\mathrm{Tr}\, A^k \mathbf{1}_E\right)\right)\right) \tag{3.1}$$

is controlled by $(1-\delta)^{2M}$.

We also have

$$\begin{aligned}\left|\sum_{k>n^{1/10}} a_k \left(\mathrm{Tr}\, A^k \mathbf{1}_E - \mathbb{E}\left(\mathrm{Tr}\, A^k \mathbf{1}_E\right)\right)\right| &\le \sum_{k>n^{1/10}} 2nc(1-\delta)^k \left(1+\frac{\log^{1+\epsilon} n}{\sqrt{n}}\right)^k \\ &\le 2nc \frac{(1-\delta)^{n^{1/10}}\left(1+\frac{\log^{1+\epsilon} n}{\sqrt{n}}\right)^{n^{1/10}}}{1-(1-\delta)\left(1+\frac{\log^{1+\epsilon} n}{\sqrt{n}}\right)} \\ &\to 0 \quad, n\to\infty.\end{aligned}$$

This implies the tail part goes to 0, even uniformly. From the finite-dimensional weak convergence to a Gaussian vector as shown in [AZ06], we have that for any fixed M, the first subsum converges in distribution to a Gaussian. M can be large enough as we want, and as (3.1) is controlled by $(1-\delta)^{2M}$. Let's now consider the *c.d.f.* for the sum of the first two terms, we denote them by X_M and Y_M,

$$\mathbb{P}(X_M < t-\epsilon) - \mathbb{P}(Y_M \ge \epsilon) \le \mathbb{P}(X_M + Y_M < t) \le \mathbb{P}(|Y_M| \ge \epsilon) + \mathbb{P}(X_M < t+\epsilon),$$

choosing M large enough, we can see $X_M + Y_M$ converge weakly to $\mathcal{N}(0, \sigma_f)$, thus the CLT holds[7]. □

4 Sinai-Soshnikov Technique

We give a brief discussion here about the combinatorial technique introduced in the original paper, and also a sketch of the proof of the main theorem, which can be seen as a strengthening of our first proof for the semicircle law.

We start with the expectation.

Theorem 4.

$$\mathbb{E}\big(\mathrm{Tr}\, A^{2s}\big) = \frac{n}{\sqrt{\pi\, s^3}}\big(1+o(1)\big) \quad \textit{as } n\to\infty,$$

uniformly in $s = o(\sqrt{n})$.

[6] Notice that here we need to use our result for growing p, while other estimates can be yielded from results for fixed p.

[7] Here we use the Slutsky's Theorem.

Since

$$\mathbb{E}\big(\mathrm{Tr}\, A^p\big) = \frac{1}{n^{p/2}} \sum_{i_0, i_1, \ldots, i_{p-1}=1}^{n} \mathbb{E}\big[\xi_{i_0 i_1}\, \xi_{i_1 i_2}\, \cdots\, \xi_{i_{p-1} i_0}\big]$$
$$= \frac{1}{n^{p/2}} \sum_{t=1}^{[\frac{p}{2}]+1} O(n^t)(\textit{Contribution of paths of } t \textit{ vertices}),$$

we need to focus on different types of *closed paths* $\mathcal{P} := i_0 \longrightarrow i_1 \longrightarrow \cdots \longrightarrow i_{p-1} \longrightarrow i_0$ of length p and their contributions to the expectation. The flavor of the argument here is similar to that for the semicircle law, but the classification of paths here is much more subtle. In the proof for the semicircle law, one basically ignores the contribution of all closed paths except the double trees, because p is fixed and the number of these paths is of smaller order than $n^{p/2+1}$. Here, this simplification doesn't work since the growth of p leads to infinitely many vertices in the graph, and the contribution of each path can't be simply controlled by one constant. However, things are not that bad since our entries are symmetrically distributed, $\mathbb{E}\big[\xi_{i_0 i_1}\, \xi_{i_1 i_2} \cdots \xi_{i_{p-1} i_0}\big] \neq 0$ only if each (unordered) *edge* of $\mathcal{P}$ is traversed even times. Such $\mathcal{P}$ are called *even paths*, and they exist only when p is even. Hence

$$\mathbb{E}[\mathrm{Tr}\, A^p] = 0 \quad \text{for odd } p,$$

and from now on we set $p = 2s$. The following multiple notions give rise to the so-called **Sinai-Soshnikov technique**.

Definition 5 (Marked steps)**.** For $\ell = 1, 2, \ldots, p$, the ℓ-th step $i_{\ell-1} \to i_\ell$ of $\mathcal{P}$ is called *marked* if this step is the odd time the edge $\{i_{\ell-1}, i_\ell\}$ appears.

Observe that step 1 is always marked, and on an even path the total number of marked steps equals the number of unmarked ones.

Definition 6 (Vertex partition)**.** For $k = 0, 1, \ldots, s$, let

$$\mathcal{N}_k(\mathcal{P}) = \big\{\, i \in \{1, \ldots, n\} : \#\text{ marked steps ending at } i = k \big\},$$

and set $n_k = \big|\mathcal{N}_k(\mathcal{P})\big|$. Then necessarily

$$\sum_{k=0}^{s} n_k = n, \qquad \sum_{k=0}^{s} k\, n_k = s. \tag{4.1}$$

We say that $\mathcal{P}$ is of *type* $(n_0, n_1, \ldots, n_s)$.

One could expect $N_k(\mathcal{P})$ for large k in the partition to be empty.

Definition 7 (Simple even path)**.** A path $\mathcal{P}$ of type

$$(n_0, n_1, \ldots, n_s) = (\, n - s,\ s,\ 0, \ldots, 0\,)$$

for which the starting vertex i_0 lies in $\mathcal{N}_0(\mathcal{P})$ is called a *simple even path.*

One can see immediately that the simple even paths is just double trees. And the truth is the main contribution is also from these paths, just as it in the semicircle law. Following this idea, we start our proof.

Sketch of proof. We shall estimate the number of closed even paths of each type and their contribution to $\mathbb{E}[\operatorname{Tr} A^p]$. Fix $(n_0, n_1, \ldots, n_s)$, we discuss different resources of freedom and how much they contribute to the limit.

First, the number of ways to decompose the set of n vertices into the $(s+1)$ subsets $\mathcal{N}_0, \mathcal{N}_1, \ldots, \mathcal{N}_s$ with $|\mathcal{N}_k| = n_k$ is

$$\frac{n!}{n_0!\, n_1! \cdots n_s!}.$$

Fix the initial vertex, whose number of choices differs for different types, and partition $\{\mathcal{N}_k\}$, freedom also comes from choices of the endpoint of each step. For the order of appearance of vertices at the marked steps, we have

$$\frac{s!}{\prod_{k=1}^{s} (k!)^{n_k}}$$

different ways to write them down. Paths now differ only by (a) the choice of which steps are marked and (b) the choice of endpoints of the unmarked steps.

Since at each time the number of marked steps is at least the number of unmarked steps, consider *#marked steps−#unmarked steps*, one get an injection (which is a bijection for simple paths) from **moments of marked steps** to **Dyck paths** of length $p = 2s$. The number of such walks is

$$C_s = \frac{(2s)!}{s!\,(s+1)!}\,.$$

With all possibilities mentioned above fixed, we only need to consider the endpoints of unmarked steps. For simple even paths, there is no such freedom (they must retrace the marked ones). Hence, the contribution of simple even paths is

$$\frac{1}{n^s} \cdot \frac{n!}{(n-s)!s!} \cdot (n-s) \cdot \frac{(2s)!}{s!(s+1)!} \cdot \frac{s!}{1!} \cdot \left(\frac{1}{4}\right)^s = \frac{n}{\sqrt{\pi s^3}}(1+o(1)). \tag{4.2}$$

For other paths in $(\, n-s,\ s,\ 0, \ldots, 0\,)$, we have a "double loop" when the second half of the path $\mathcal{P}$ repeats the first one. Their contribution is $\frac{n-s}{s}$ times smaller because of different choices for the initial point, and can be neglected.

If $n_1 < s$, the choice of the end points at the unmarked steps from the vertices of type $\mathcal{N}_k, k \geq 2$ may be non-unique. After careful discussion, one can say we have at most $2k$ possibilities for the right end of these steps.

For the paths of $(\, n_0, n_1, \ldots, n_s\,)$ type

$$\left|\mathbb{E}[\xi_{i_0 i_1} \ldots \xi_{i_{p-1} i_0}]\right| \leq \prod_{k=1}^{s} (\text{const} \cdot k)^{k \cdot n_k},$$

their contribution can be estimated from above by

$$\begin{aligned} &\frac{1}{n^s} \cdot \frac{n!}{n_o! n_1! \ldots n_s!} \cdot n \cdot \frac{(2s)!}{s!(s+1)!} \cdot \frac{s!}{\prod_{k=2}^{s} (k!)^{n_k}} \cdot \prod_{k=2}^{s} (2k)^{k \cdot n_k} \cdot \prod_{k=2}^{s} (\text{const} k)^{k n_k} \\ &\leq n \cdot \frac{(2s)!}{s!(s+1)!} \cdot \frac{1}{4^s} \cdot \left[\frac{n(n-1)\ldots(n_0+1)}{n^s \cdot n_1! \cdots n_s!} \cdot \frac{s!}{\prod_{k=2}^{s} (ke^{-1})^{k n_k}} \cdot \prod_{k=2}^{s} \left(2 \text{ const } k^2\right)^{k \cdot n_k} 4^s \right]. \end{aligned}$$

Under proper estimate, the sum of the last expression over all non-negative integers $n_2, n_3, \ldots$, n_k such that

$$0 < \sum_{k=2}^{s} k \cdot n_k \leq s$$

is not greater than

$$n \cdot \frac{(2s)!}{s!(s+1)!} \cdot \left(\frac{1}{4}\right)^s \cdot \left(\exp\left(\sum_{k=2}^{s} \frac{(8e \cdot \text{ const } \cdot k \cdot s)^k}{n^{k-1}}\right) - 1\right)$$

Since for $s \ll n^{1/2}$

$$\sum_{k=2}^{s} \frac{(8e \text{ const } k \cdot s)^k}{n^{k-1}} = O\left(\frac{s^2}{n}\right) = o(1).$$

One could realize that now main contribution is only from (4.2). □

The takeaway here is the following proposition, which will also be used in next steps.

Corollary 8. *The main contribution to the number of all even paths of length p on the set of n vertices $\{1, 2, \ldots, n\}$ where $p = o\left(n^{1/2}\right)$ as $n \to \infty$ is given by simple even paths, i.e.*

$$\frac{\#_{n,p} \textit{ simple even paths}}{\#_{n,p} \textit{ even paths}} \underset{n\to\infty}{\longrightarrow} 1.$$

5 Following Steps

To prove the central limit theorem, one now analyzes the variance and then higher moments of $\operatorname{Tr} A^p$. By considering pairs of paths with a common edge and whose union has even multiplicity of each edge, which we call them *correlated*, one can prove:

Theorem 9. *Let $p = o(\sqrt{n})$. Then $\operatorname{Var}(\operatorname{Tr} A^p) \leq$ const for all n and $\operatorname{Var}(\operatorname{Tr} A^p) \to \frac{1}{\pi}$ as $n \to \infty, p \to \infty, \frac{p}{\sqrt{n}} \to 0$.*

Sketch of proof. Since

$$\begin{aligned}\operatorname{Var}(\operatorname{Tr} A^p) &= \mathbb{E}(\operatorname{Tr} A^p)^2 - (\mathbb{E} \operatorname{Tr} A^p)^2 \\ &= \sum_{i_0,i_1,\ldots,i_{p-1}=1}^{n} \sum_{j_0,j_1,\ldots,j_{p-1}=1}^{n} \frac{1}{n^p} \\ &\quad \cdot \left(\mathbb{E} \prod_{\ell=1}^{p} \xi_{i_{\ell-1}i_\ell} \cdot \prod_{m=1}^{p} \xi_{j_{m-1}j_m} - \mathbb{E} \prod_{\ell=1}^{p} \xi_{i_{\ell-1}i_\ell} \cdot \mathbb{E} \prod_{m=1}^{p} \xi_{j_{m-1}j_m}\right)\end{aligned}$$

Terms are nonzero only if the pairs are correlated. The goal is to show that the main contribution to the number of correlated pairs and to the variance is due to **simply correlated pairs**, i.e., each edge appears in the union of the paths only twice.

To show this, one construct a map from correlated pairs to even paths of length 2p-2 and argues the number of their preimages, see Figure 1. To study the number of ways to recover a pair of paths from the even path, one can define the **joint edge** to be the first edge along

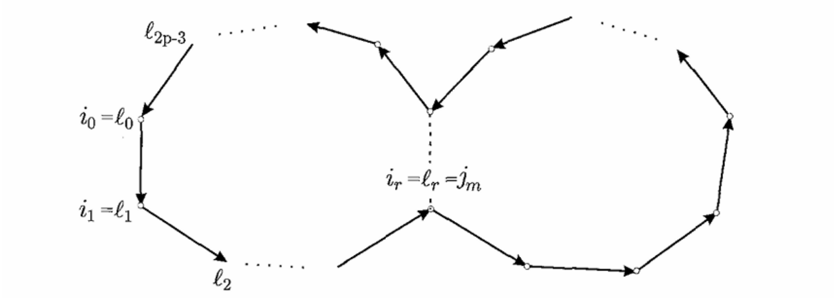

Figure 1: Construction of even paths of length 2p-2.

the first path that coincides with edge in the second path, and then relate the possibility of choosing a joint edge from an even path to a *Dyck path*.

One will show that there are at most $2p$ ways to choose the starting point and direction for the second path from any legal even path, and

$$2^{2p-2} \cdot \frac{2}{\pi} \cdot \frac{1}{p} \cdot (1 + o(1))$$

ways to pick the legal *Dyck path* and the joint edge.

As a result, the main order of the number of correlated pairs equals to the number of simply correlated pairs and is

$$\frac{1}{\pi} \cdot n^p \cdot 2^{2p} \cdot (1 + o(1)),$$

where the n^p comes from choices of vertices in the even path.

If one takes into account the weights

$$\mathbb{E} \prod_{\ell=1}^{p} \xi_{i_{\ell-1} i_\ell} \cdot \prod_{m=1}^{p} \xi_{j_{m-1} j_m} - \mathbb{E} \prod_{\ell=1}^{p} \xi_{i_{\ell-1} i_\ell} \cdot \mathbb{E} \prod_{m=1}^{p} \xi_{j_{m-1} j_m},$$

ascribed to the correlated paths gives

$$\operatorname{Var}(\operatorname{Tr} A^p) \underset{n\to\infty}{\longrightarrow} \frac{1}{\pi}, \quad 1 \ll p \ll \sqrt{n}.$$

□

Finally, we use **the method of moments** to prove the central limit theorem. One needs to consider the asymptotics of higher moments. The main idea is rather straightforward and analogous to what we have done for the variance. Since

$$\mathbb{E}(\operatorname{Tr} A^p - \mathbb{E} \operatorname{Tr} A^p)^L = \frac{1}{n^{\frac{pL}{2}}} \cdot \mathbb{E} \prod_{m=1}^{L} \left(\sum_{i_0^{(m)}, i_1^{(m)}, \ldots, i_{p-1}^{(m)}} \left(\prod_{r=1}^{p} \xi_{i_{r-1}^{(m)}, i_r^{(m)}} - \mathbb{E} \prod_{r=1}^{p} \xi_{i_{r-1}^{(m)}, i_r^{(m)}} \right) \right),$$

one then needs to consider L coupled closed paths

$$P_m = \{ i_0^{(m)} \to i_1^{(m)} \to \cdots \to i_{p-1}^{(m)} \to i_p^{(m)} \}, \quad m = 1, \ldots, L.$$

To simplify this problem, one defines the term *Cluster of correlated paths* and constructs an even path for each correlated cluster. The argument left now will be almost the same as what we did for the variance. One can obtain the CLT for finite-dimensional distributions in the same way. See [SS98] for more details.

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