Montgomery, Dyson, and the Suprising Link Between Random Matrices and the Riemann Zeta Function

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April 28, 2025

1 Montgomery's Encounter with Dyson

Hugh Montgomery was working in analytic number theory, studying the zeros of the Riemann Zeta function. He had been able to develop an impressive asymptotic formula (which we will see later) that displayed the pair correlations between the zeros. In a happenstance conversation with Freeman Dyson, a physicist and pioneer in Random Matrix Theory, he mentioned he had obtained this. Without seeing the result, Dyson asked whether the formula took the following form:

$$\int_a^b \left(1 - \left(\frac{\sin \pi x}{\pi x}\right)^2\right) dx$$

Incredibly, Dyson was spot-on. But how did he possibly guess such a behavior? It turns out that this same expression appears in the asymptotics of random unitary matrices. It is our goal to formulate and investigate this remarkable parallel.

2 Circular Unitary Ensemble

To talk about these matrix asymptotics, it is helpful to introduce another common matrix ensemble, namely, the Circular Unitary Ensemble (CUE). These are random matrices found by uniformly sampling the Haar measure on the group of unitary matrices. Locally, CUE and GUE have the same statistics (up to proper formulation), but CUE is completely uniform; in particular it does not have an edge. The eigenvalues of a CUE matrix fall on the complex unit circle. Letting λ and λ' be eigen*angles* of a CUE matrix (real numbers such that $e^{i\lambda}, e^{i\lambda'}$ are eigenvalues), we have the following asymptotic expression:

$$\lim_{N\to\infty} \mathbb{E}\left[N^{-1} \sum_{\lambda,\lambda'} \mathbb{1}_{[a,b)}\left((\lambda - \lambda')\frac{N}{2\pi}\right)\right] = \mathbb{1}_{[a,b]}(0) + \int_a^b \left(1 - \left(\frac{\sin \pi x}{\pi x}\right)^2\right) dx$$

Where N is the number of eigenvalues of the matrix (See [5]). With different scaling to account for the semicircle distribution, we could obtain an extremely similar statement for the eigen*values* of GUE matrices. The integrand on the right hand side is called the pair correlation function.

3 The Riemann Zeta Function

3.1 Definition

We begin by establishing some of the basics of the Zeta function so that we can make sense of pair correlations for its zeros. We define $\zeta : (1, \infty) \to \mathbb{R}$,

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}$$

This function, and its analytic extension to $\mathbb{C} \backslash \{1\}$ are known as the Riemann Zeta Function. One derivation of the extension leverages the Gamma Function (which is meromorhpic) to obtain the following:

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{x^{s-1}}{1 - e^x}\, dx$$

This equation defines an analytic function on $\mathbb{C} \setminus \{1\}$ with a simple pole at 1 thanks to the known analytic extension of the Gamma Function.

3.2 Zeta zeros

To study the zeros of the extended ζ function, we define a related function: $\xi : \mathbb{C} \setminus \{0, 1\} \to \mathbb{C}$,

$$\xi(s) = \pi^{-s/2} \Gamma(s/2) \zeta(s)$$

This function is analytic on its domain, and it may be checked that $\xi(s) = \xi(1 - s)$.

As Γ has poles at the negative integers, we see that ζ must trivially have zeros at the negative even numbers (in order for ξ to remain analytic). We also have that the ζ zeros are symmetrical about the real line, since $\zeta(\overline{s}) = \overline{\zeta(s)}$, as can be checked from the integral representation of ζ. Additionally, the non-trivial zeros are symmetric about $\mathrm{Re}\ s = 1/2$, since $\zeta(s) = 0$ if and only if $0 = \xi(s) = \xi(1 - s)$, which is true if and only if $\zeta(1 - s) = 0$. As ζ is nonvanishing for $\mathrm{Re}\ s > 1$, we have that the nontrivial zeros must all be contained in the critical strip $\{z \in \mathbb{C} : 0 \leq \mathrm{Re}\ s \leq 1\}$.

Every known nontrivial zero falls on the critical line $\mathrm{Re}\ s = 1/2$. However, it is unknown whether all such zeros fall on this line. This is precisely the crux of the Riemann Hypothesis:

Conjecture (Riemann): *Every nontrivial zero of the zeta function has real part* $1/2$.

We will do as many authors already do and assume this conjecture to be true, unless otherwise noted. Before proceeding, we state a result seen in [2] about the asymptotics of these zeros that is true independently of the Riemann Hypothesis:

Lemma: *For* $T > 0$*, let* $N(T)$ *be the number of zero zeros in the critical strip with imaginary part between 0 and* T*. Then, as* $T \to \infty$,

$$N(T) = \frac{T}{2\pi} \log T + o(T \log(T))$$

4 Montgomery's Conjecture

4.1 Montgomery's Theorem

Denote by $\rho = \frac{1}{2} + i\gamma$ a nontrivial zero of ζ. We are interested in getting some sort of pair correlation for these γ. To study this, we introduce a carefully chosen function. For $\alpha \in \mathbb{R}, T \geq 2$, and γ, γ' zeros define

$$F(\alpha) = F(\alpha, T) = \left(\frac{T}{2\pi}\log T\right)^{-1} \sum_{0<\gamma,\gamma' \leq T} T^{i\alpha(\gamma-\gamma')} w(\gamma - \gamma') \tag{1}$$

Where $w(u) = \frac{4}{4+u^2}$ is a weight function and the coefficient of the sum reflects the asymptotics of the Zeta zeros seen in Lemma 1. One of the advantages of F is that it interacts nicely when integrated against a Fourier transform:

$$\int_{\mathbb{R}} F(\alpha)\hat{r}(\alpha)\, d\alpha = \left(\frac{T}{2\pi}\log T\right)^{-1} \sum_{0<\gamma,\gamma' \leq T} r\left((\gamma - \gamma')\frac{\log T}{2\pi}\right) w(\gamma - \gamma') \tag{2}$$

Where the right hand side of the equation provides us the same sort of sum we had for the matrix asymptotics. This equivalence can be easily verified by noting that $T^{i\alpha(\gamma-\gamma')} = \exp(2\pi i\alpha\frac{\gamma-\gamma'}{2\pi}\log T)$ and passing the sum outside of the integral. The integral expression will allow us to study the asymptotics of the sum on the right hand side. To do this, we use Montgomery's Theorem, a result in analytic number theory:

Theorem: (Montgomery, 1972/73) *Let F be as above. Then for all α, $F(\alpha) \in \mathbb{R}$ and $F(\alpha) = F(-\alpha)$, and for every $\varepsilon > 0$, there exists $T_0 = T_0(\varepsilon)$ such that if $T > T_0$, then $F(\alpha) \geq -\varepsilon$ for all α. Finally, for fixed $\alpha \in [0,1)$, we have*

$$F(\alpha) = (1 + o(1))T^{-2\alpha}\log T + \alpha + o(1) \tag{3}$$

uniformly for $0 \leq \alpha \leq 1 - \varepsilon$ as $T \to \infty$.

An important corollary of this theorem is the main point of interest for our discussion. **Suppose that r is a function whose Fourier transform is supported inside $(-1,1)$.** Let then us combine the result of (3) with the formulation in (2):

$$\left(\frac{T}{2\pi}\log T\right)^{-1} \sum_{0<\gamma,\gamma' \leq T} r\left((\gamma - \gamma')\frac{\log T}{2\pi}\right) w(\gamma - \gamma') \sim \int_{\mathbb{R}} (T^{-2|\alpha|}\log T + |\alpha|)\hat{r}(\alpha)\, d\alpha$$

Using linearity and applying a change of variables $\alpha \mapsto \alpha \log T$ in the first integral, we find that the asymptotic expression is equal to:

$$\int_{\mathbb{R}} e^{-2|\alpha|}\hat{r}\left(\frac{\alpha}{\log T}\right) d\alpha + \int_{\mathbb{R}} |\alpha|\hat{r}(\alpha)\, d\alpha \tag{4}$$

We assess these two terms separately. Our goal is to obtain a limiting expression in terms of r and not $\hat{r}$. For the first term, apply the dominated convergence theorem to find that:

$$\lim_{T\to\infty} \int_{\mathbb{R}} e^{-2|\alpha|}\hat{r}\left(\frac{\alpha}{\log T}\right) d\alpha = \int_{\mathbb{R}} e^{-2|\alpha|}\hat{r}(0)\, d\alpha$$

$$= \hat{r}(0) = \int_{\mathbb{R}} r(x)\, dx$$

by evaluating the integral on α and then applying the Fourier Inversion Formula.

For the second term, we split the integral into two parts so we can apply a known Fourier Transform:

$$\int_{\mathbb{R}} |\alpha| \hat{r}(\alpha)\, d\alpha = \int_{-1}^{1} \hat{r}(\alpha)\, d\alpha - \int_{-1}^{1} (1 - |\alpha|)\hat{r}(\alpha)\, d\alpha$$

For the first integral, we apply the Fourier Invesion Formula to obtain $r(0)$. For the second term, we can apply Fubini's Theorem to swap the Fourier transform from $r(x)$ to $(1-|x|)\mathbb{1}_{[-1,1]}$. It is an exercise to compute that the Fourier transform of the latter is precisely $\left(\frac{\sin \pi x}{\pi x}\right)^2$. So overall, this term becomes

$$r(0) - \int_{\mathbb{R}} \left(\frac{\sin \pi x}{\pi x}\right)^2 r(x)\, dx$$

Combining all of these, we have that (4) is equivalent to

$$r(0) + \int_{\mathbb{R}} \left(1 - \left(\frac{\sin \pi x}{\pi x}\right)^2\right) r(x)\, dx$$

Thus, overall we have that

$$\left(\frac{T}{2\pi} \log T\right)^{-1} \sum_{0<\gamma,\gamma' \leq T} r\left((\gamma - \gamma')\frac{\log T}{2\pi}\right) w(\gamma - \gamma') \underset{T\to\infty}{\to} r(0) + \int_{\mathbb{R}} \left(1 - \left(\frac{\sin \pi x}{\pi x}\right)^2\right) r(x)\, dx$$

4.2 Montgomery's Conjecture

The integrand above contains the $n = 2$ case of the sine kernel that we saw in the eigenvalue process for GUE. Moreover, Montgomery hypothesized that the results above would hold even for functions r whose Fourier transform has unbounded support. In particular, he conjectured that it might work for $r(\alpha) = \mathbb{1}_{[a,b]}$ for $a < b$ real numbers. In this case, we would obtain the following asymptotic statement:

$$\left(\frac{T}{2\pi} \log T\right)^{-1} \sum_{0<\gamma,\gamma' \leq T} \mathbb{1}_{[a,b]} \left((\gamma - \gamma')\frac{\log T}{2\pi}\right) w(\gamma-\gamma') \underset{T\to\infty}{\to} \mathbb{1}_{[a,b]}(0) + \int_{a}^{b} \left(1 - \left(\frac{\sin \pi x}{\pi x}\right)^2\right) dx$$

The indicator function on the right hand side may be removed by requiring that $0 \notin [a, b]$ (ie that $\gamma \neq \gamma'$). Essentially, this suggests that the pair correlations of the Zeta zeros are the same as the pair correlations of the eigenvalues for a unitary random matrix. This lends credence to the Hilbert-Pólya Conjecture, which proposes that the zeros of ζ are precisely the eigenvalues of some linear operator, and that the Riemann Hypothesis is equivalent to such operator being self-adjoint [1].

Thus, this connection between random matrix theory and the Riemann Zeta function is not only a nice result to look at, it also helps peel back another potential layer in the search for a proof of the most elusive problem in mathematics.

5 References

[1] J. Peca-Medlin, *An Approach to the Riemann Hypothesis through Random Matrix Theory*, 2018.

[2] K. Prodomidis (advised by Lucas Benigni), *The Riemann Zeta Function and Random Matrix Theory*.

[3] *Analytic Continuation of the Riemann Zeta Function*, University of Oklahoma, 2010.

[4] H. L. Montgomery, *The Pair Correlation of Zeros of the Zeta Function*, Analytic number theory (Proc. Sympos. Pure Math., Vol. XXIV, St. Louis Univ., St. Louis, Mo., 1972), Amer. Math. Soc., Providence, R.I., 1973, pp. 181-193.

[5] E. Meckes, *The Random Matrix Theory of the Classical Compact Groups*

[6] L. García, *A Brief Introduction to Montgomery Conjecture*, Sergio Alboleda University, 2014.

6 Appendix

The Riemann Zeta Function is also given by the Euler Product formula:

$$\zeta(s) = \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}}$$

6.1 Analytic Continuation

This function can be extended to a complex domain; if Re $s > 1$, we can simply use the original series definition to obtain a well-defined analytic function. From there, it is, possible to analytically extend this function to $\mathbb{C} \setminus \{1\}$.

We can leverage the Gamma Function in order to perform this extension, as seen in [1]. Recall that the Gamma Function is given by:

$$\Gamma(s) = \int_0^\infty e^{-x} x^{s-1} \, dx$$

which extends to a nonvanishing meromorphic function on $\mathbb{C}$ with poles on $-\mathbb{N}$. Take any $n \in \mathbb{N}$ and apply the change of variables $x \mapsto x/n$. Then,

$$\Gamma(s) = \int_0^\infty (nx)^{s-1} e^{-nx} \, (ndx) = n^s \int_0^\infty x^{s-1} e^{-nx} \, dx$$

Using this form of the Gamma function and the Monotone Convergence Theorem, we can write:

$$\begin{aligned} \Gamma(s)\zeta(s) &= \sum_{n=1}^\infty n^{-s}\Gamma(s) = \sum_{n=1}^\infty \int_0^\infty x^{s-1} e^{-nx} \, dx \\ &= \int_0^\infty x^{s-1} \sum_{n=1}^\infty (e^{-x})^n \, dx \\ &= \int_0^\infty x^{s-1} \frac{1}{1 - e^x} \, dx \end{aligned}$$

Therefore, we have

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{x^{s-1}}{1 - e^x} \, dx$$

While we have defined this for Re $s > 1$, this equation defines an analytic function on $\mathbb{C} \setminus \{1\}$ with a simple pole at 1 thanks to the known analytic extension of the Gamma Function.