Dimer bijections, Aztec triangles, and spanning forests

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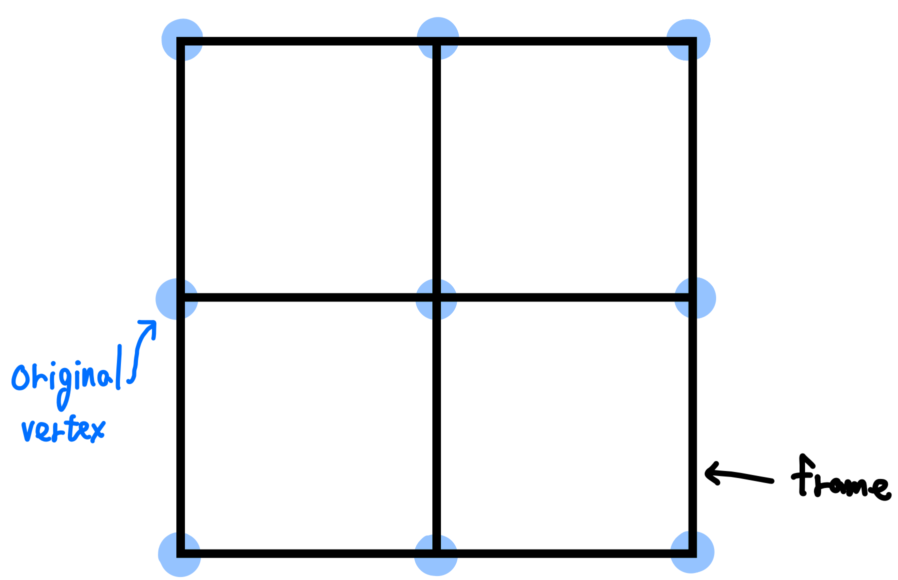
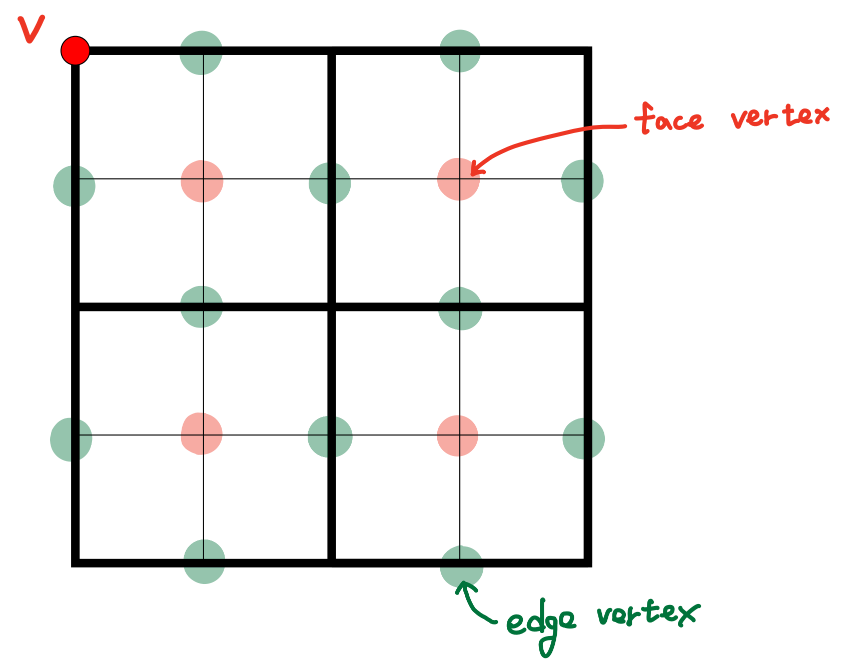
Based on a joint work with Mihai Ciucu (Indiana)

Temperley's bijection

Let R_m be a $m \times m$ square graph.
Consider R_m and take a "dual refinement" of it, which is R_{2m-1}.
Then, choose a vertex v of R_m, which is also a vertex of R_{2m-1}, from the boundary.





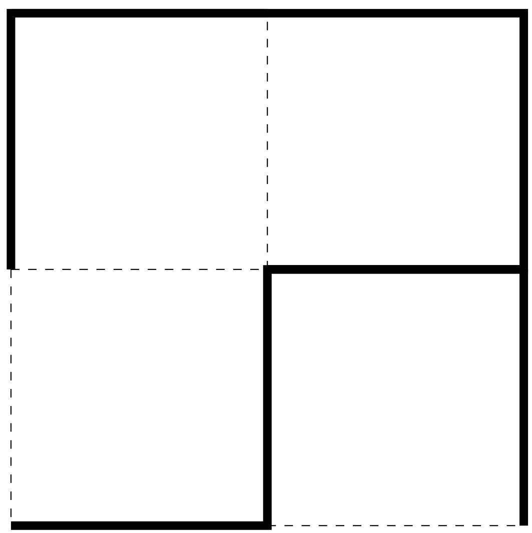
* original/face/edge vertices / Frame .

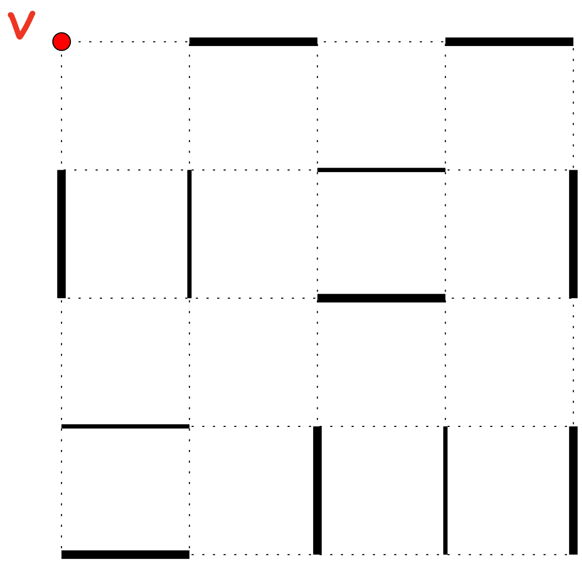
Temperley's bijection

For a graph G, let $T(G)$ and $\mathcal{M}(G)$ be sets of spanning trees and perfect matchings of it, respectively.

Temperley's bijection

There is a (natural) bijection between $T(R_m)$ and $\mathcal{M}(R_{2m-1} \setminus v)$.



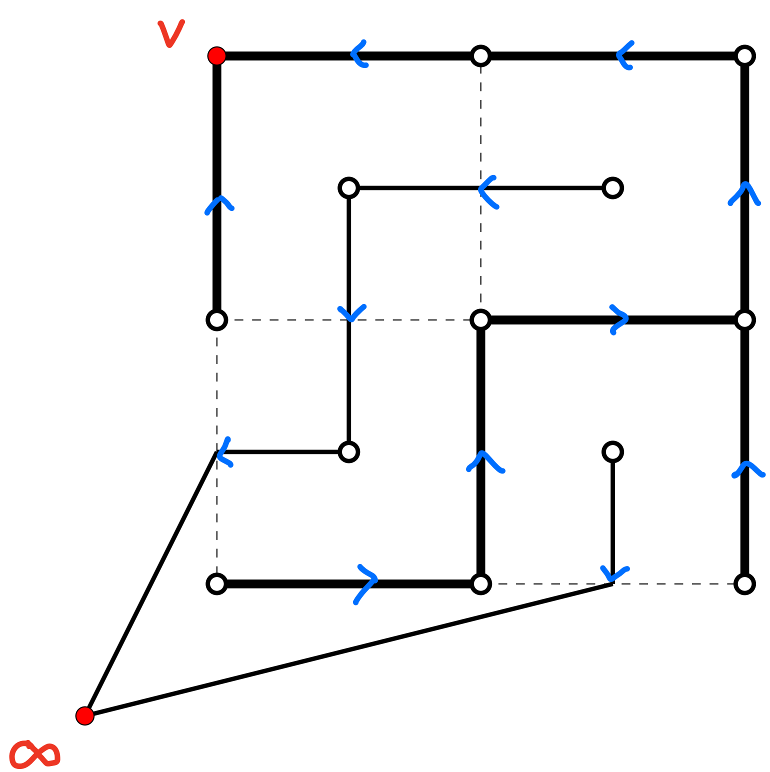


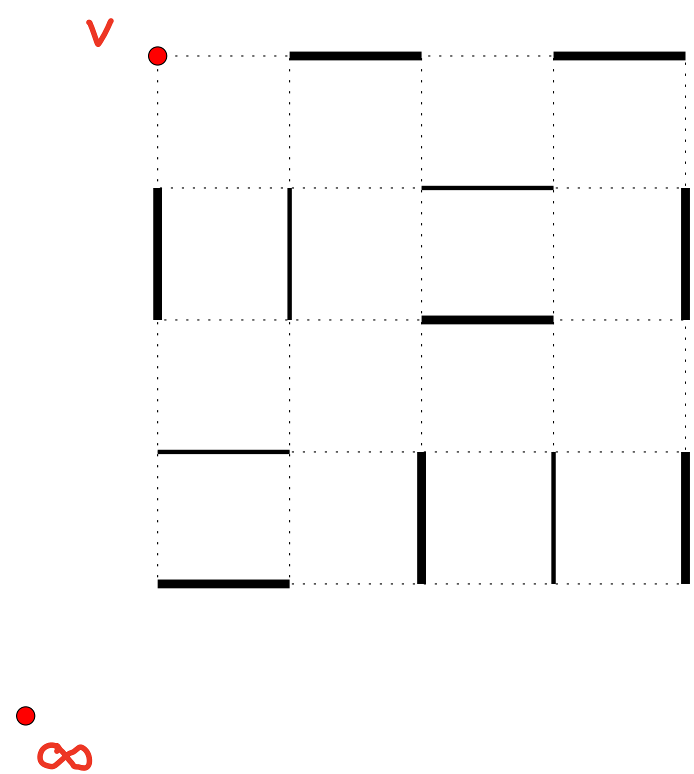
Corollary: $|\mathcal{M}(R_{2m-1} \setminus v)|$ does not depend on v.

Temperley's bijection: proof sketch

From a spanning tree to a perfect matching:

1. Also consider "dual spanning tree".
2. Regard both spanning/dual spanning trees as rooted trees (at v and ∞).
3. Take "tail-half" edges.

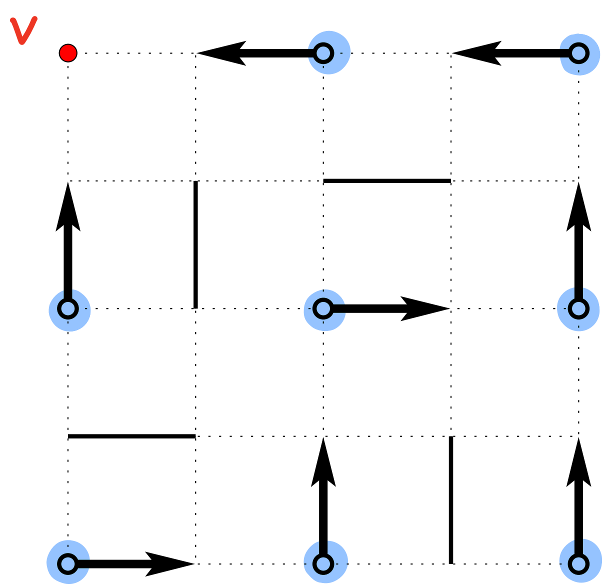


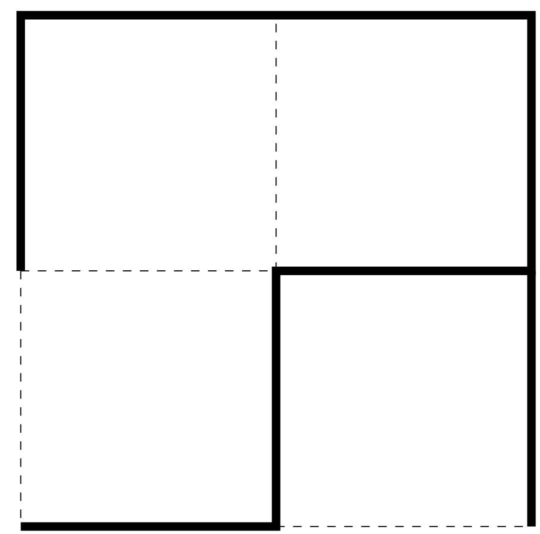


Temperley's bijection: proof sketch

From a perfect matching to a spanning tree:

1. Only look at edges that are on the frame. Note that they contain original vertices.
2. Regard them as directed edges starting from their original vertices.
3. "Double" all those edges.

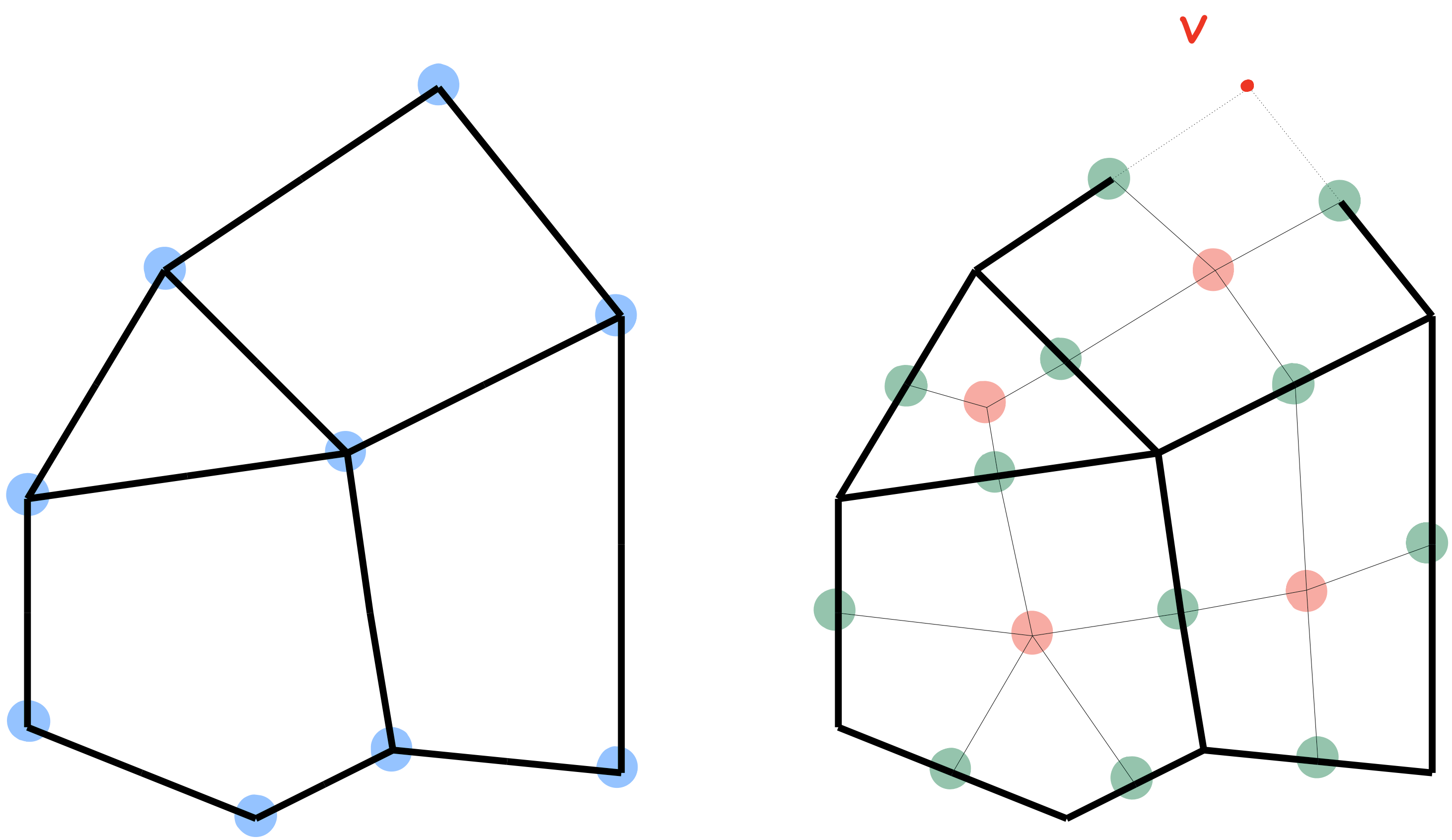




* ~~We will see this construction again at the end of the talk~~ (if time permit).

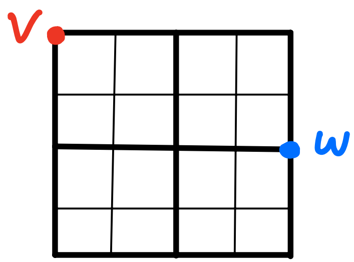
Comment about Temperley's bijection

It is known that this correspondence is not limited to square graphs, but valid for generic graphs [Lovász] and even valid for weighted and directed graphs [Kenyon, Propp, Wilson].



A related dimer bijection

Let us choose two original vertices v and w from R_m that are on the boudnary. Temperley's bijection says we have

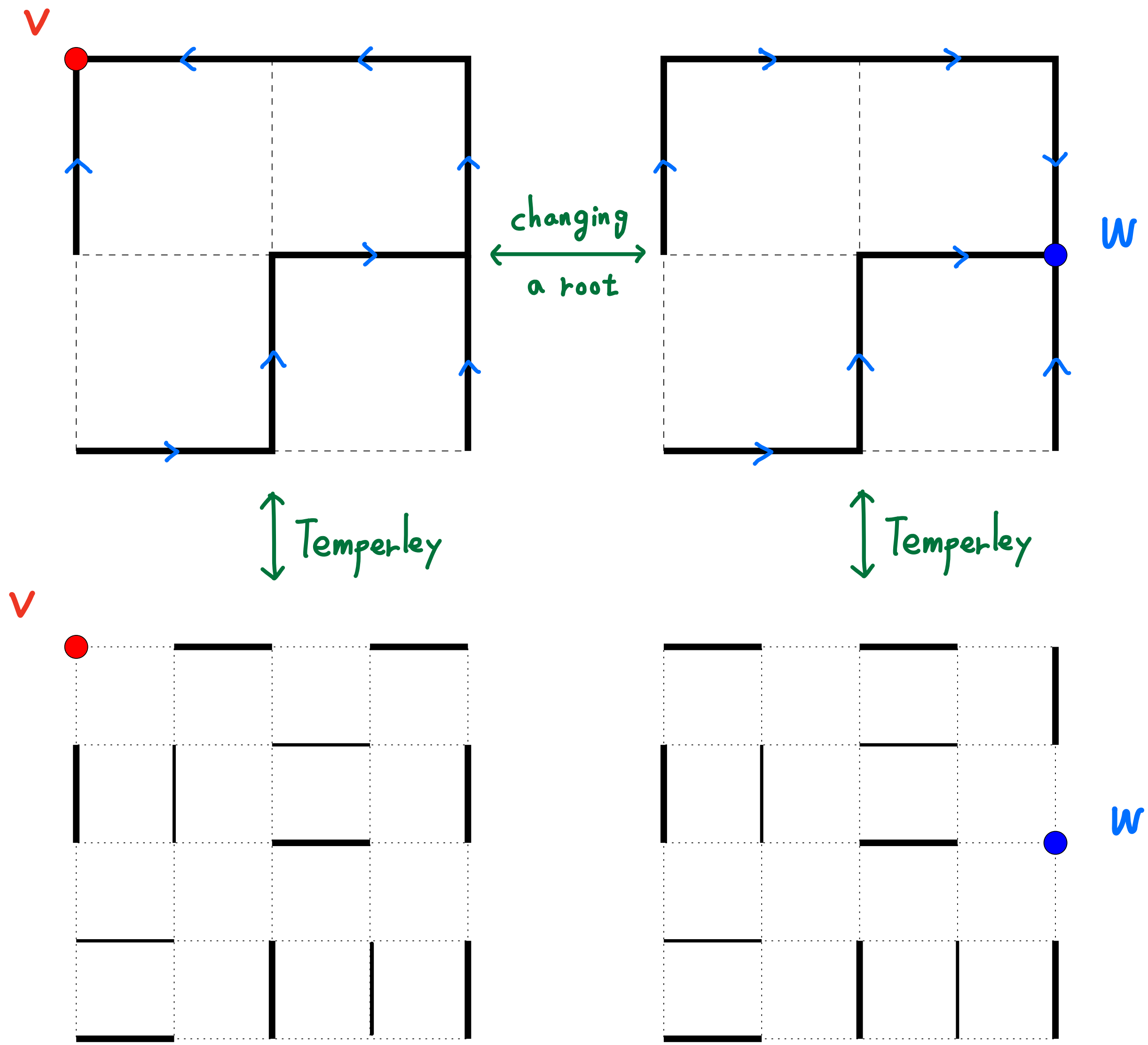


1. a bijection between $T(R_m)$ and $\mathcal{M}(R_{2m-1} \setminus v)$ and
2. a bijection between $T(R_m)$ and $\mathcal{M}(R_{2m-1} \setminus w)$.

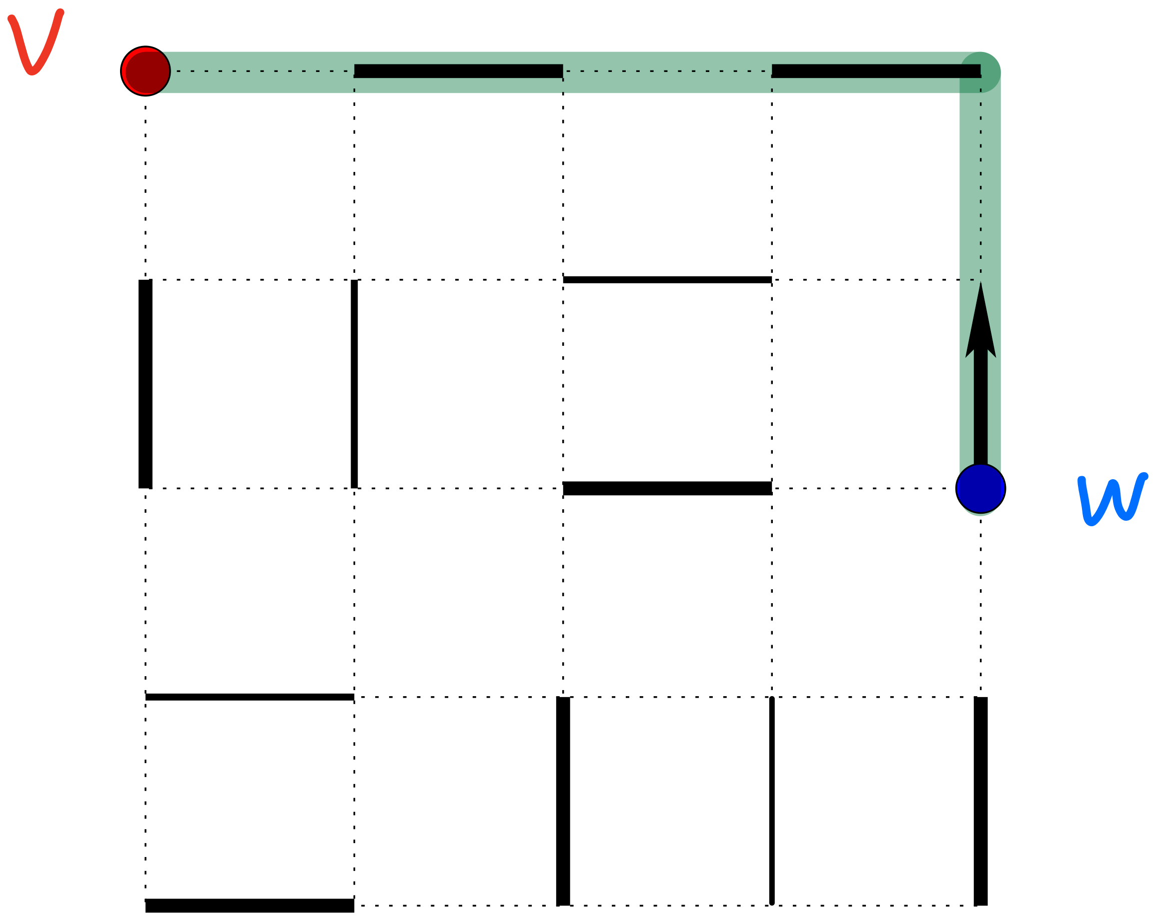
Thus, by composing these two bijections, we can construct a bijection between $\mathcal{M}(R_{2m-1} \setminus v)$ and $\mathcal{M}(R_{2m-1} \setminus w)$ "via" $T(R_m)$.

[Kenyon, Wilson, Propp] explained how one can map $\mathcal{M}(R_{2m-1} \setminus v)$ to $\mathcal{M}(R_{2m-1} \setminus w)$ (and vice versa) without going through $T(R_m)$: "gliding"

A related dimer bijection

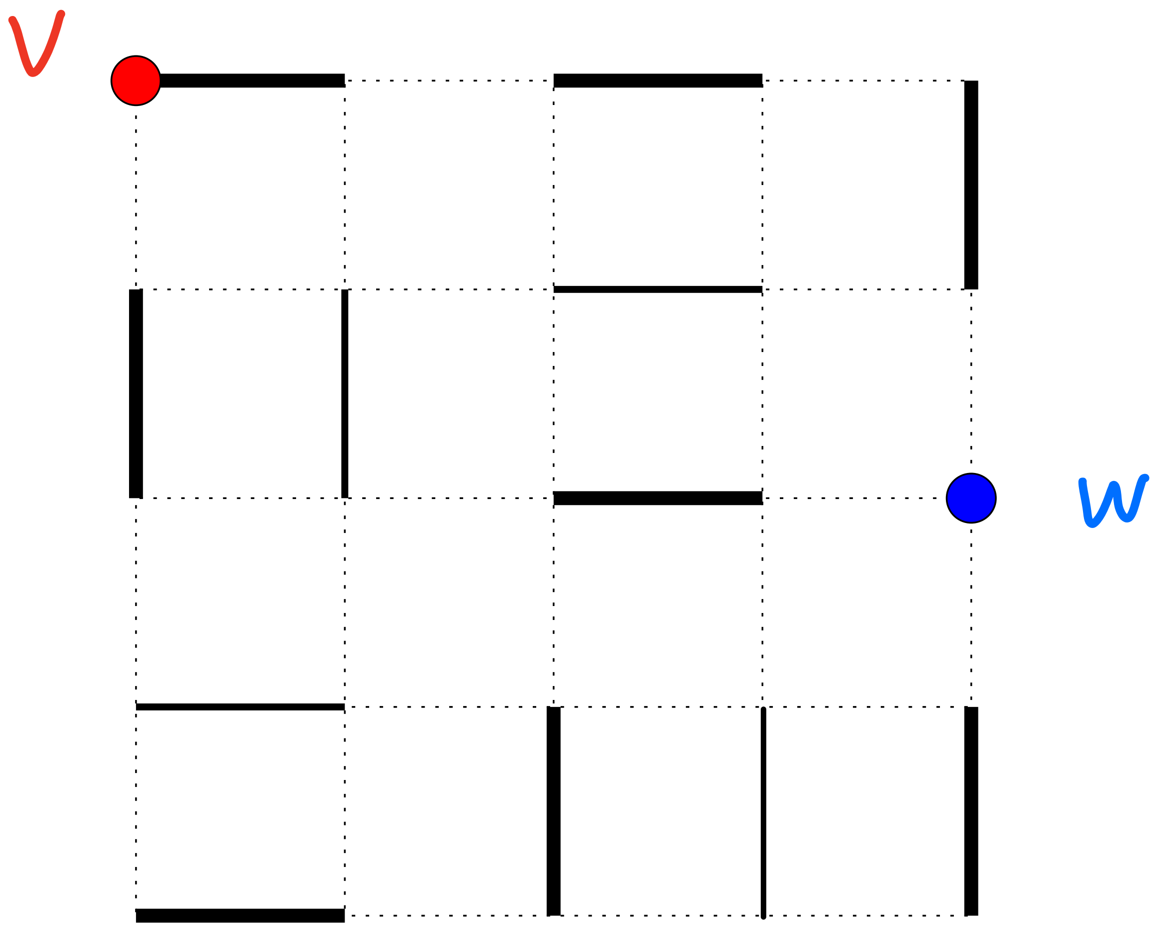


Gliding (1/2)



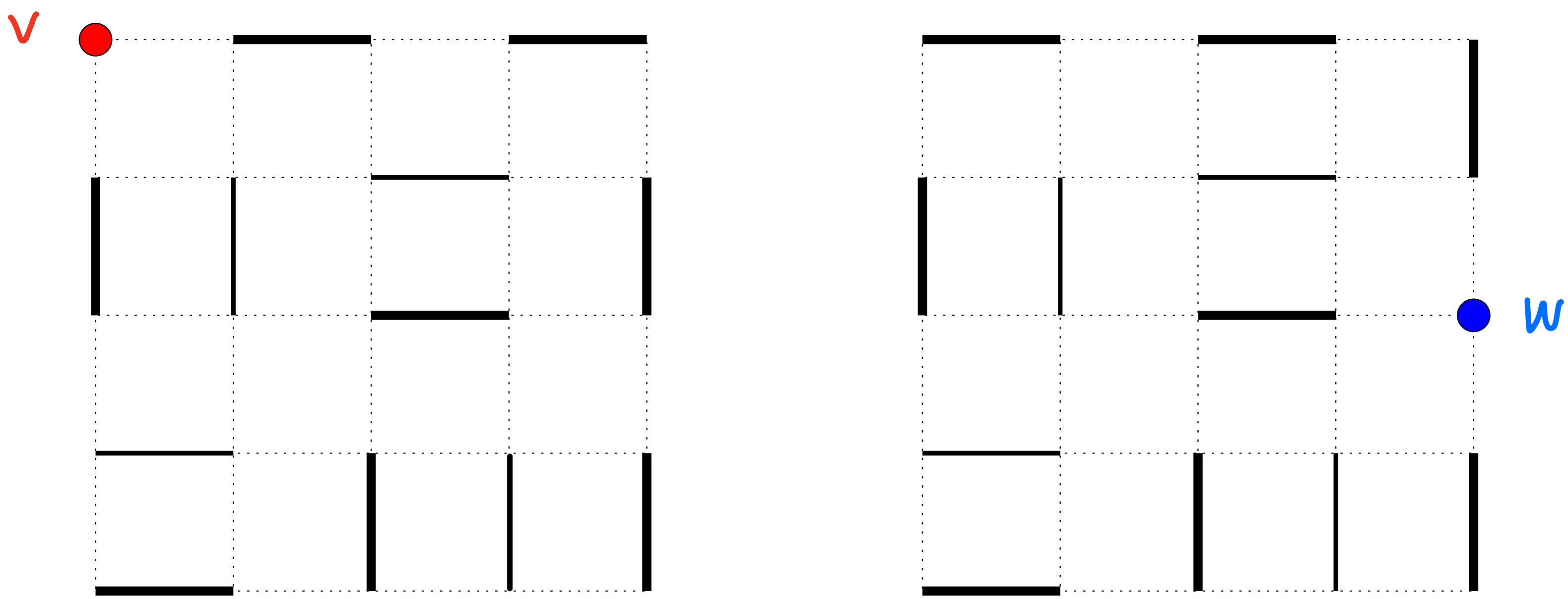
* move twice the distance along edges.

Gliding (2/2)

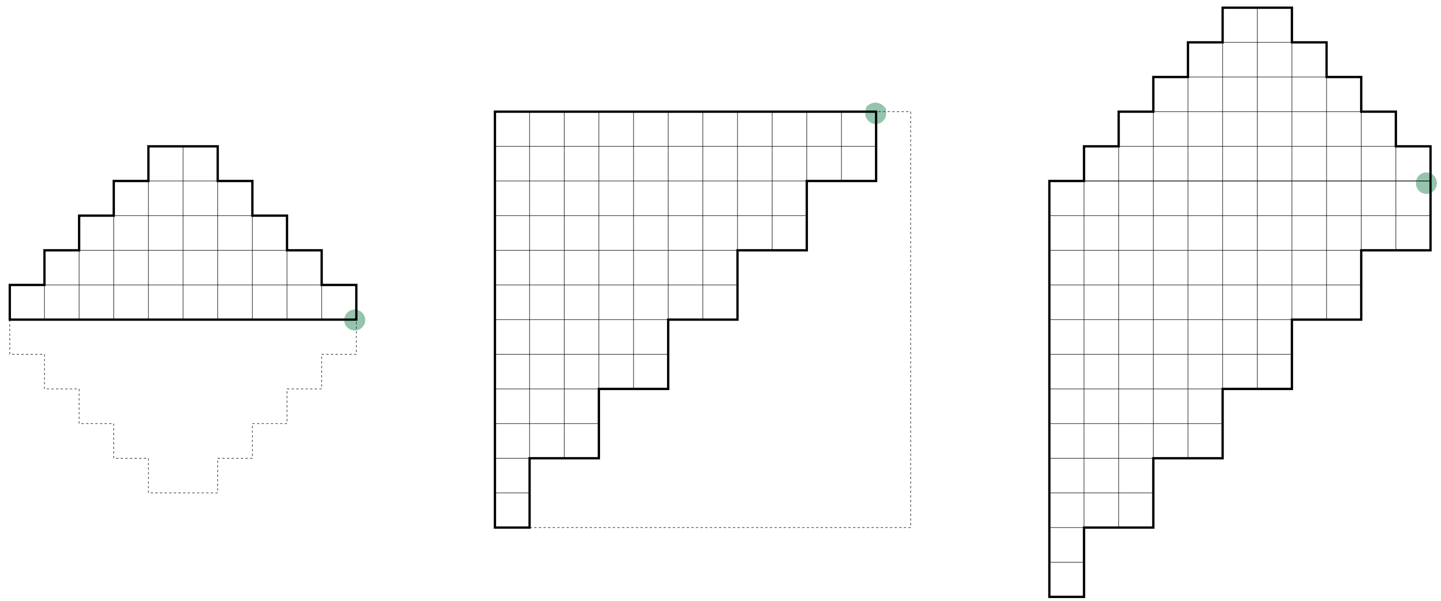


A related dimer bijection

Now, we can see the bijection without the help of the corresponding spanning tree.



Construction of Di Francesco's Aztec triangle and a conjecture



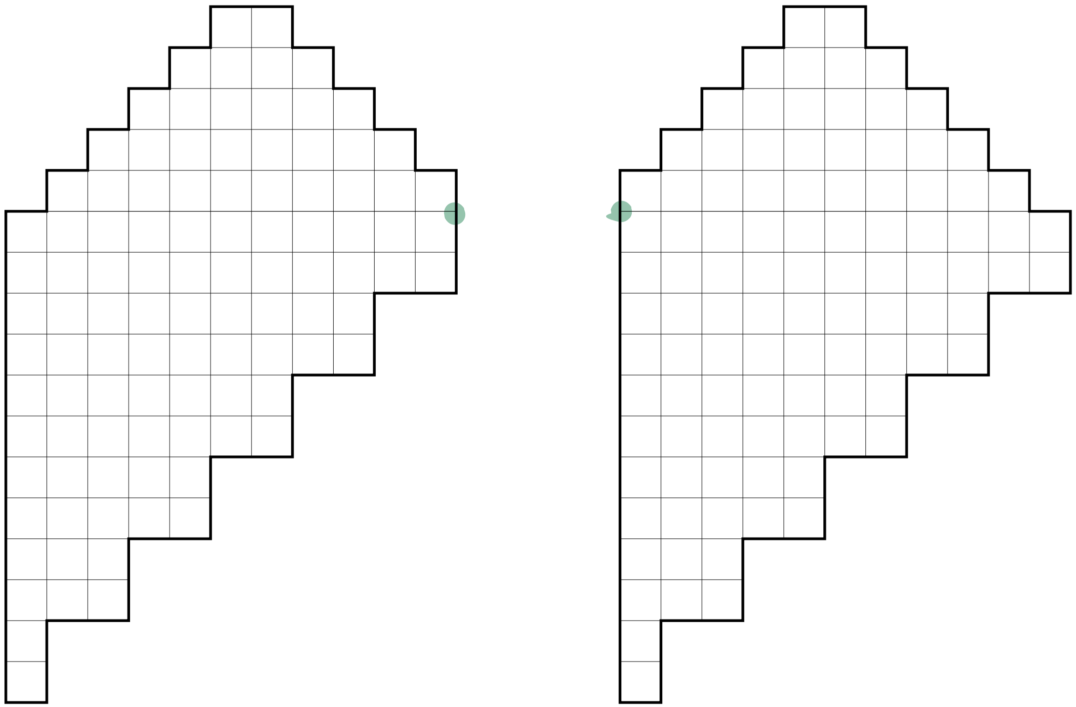
$$\frac{1}{2}AD(n-1)+\frac{1}{2}(\text{size } 2n \text{ chessboard}) \Rightarrow T_n$$

Di Francesco conjectured the number of domino tilings of T_n:

$$2^{n(n-1)/2}\prod_{j=0}^{n-1}\frac{(4j-2)!}{(n+2j+1)!}.$$

Resolution of the conjecture and a question.

The conjecture was resolved (by Koutschan / Corteel, Huang, and Krattenthaler / Koutschan, Krattenthaler, and Schlosser)

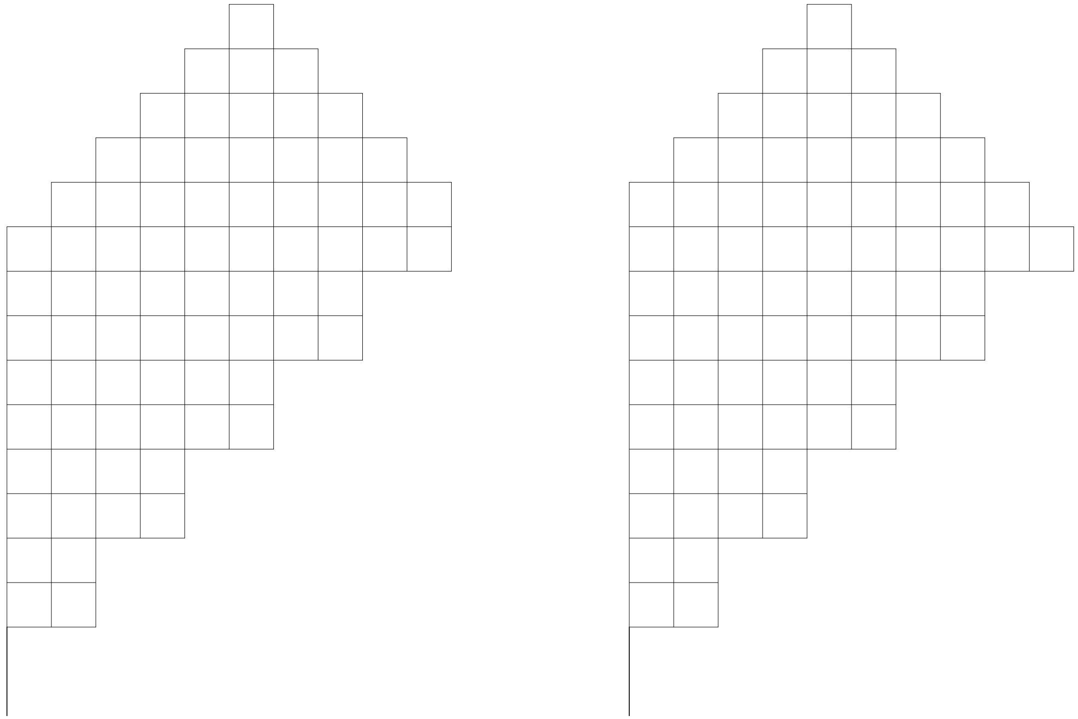


Aztec triangles $\mathcal{T}_n$ and $\mathcal{T}'_n$ of order 6.

Open question [Corteel, Huang, Krattenthaler]

Find a bijection between sets of domino tilings of $\mathcal{T}_n$ and $\mathcal{T}'_n$.

From domino tilings to perfect matchings



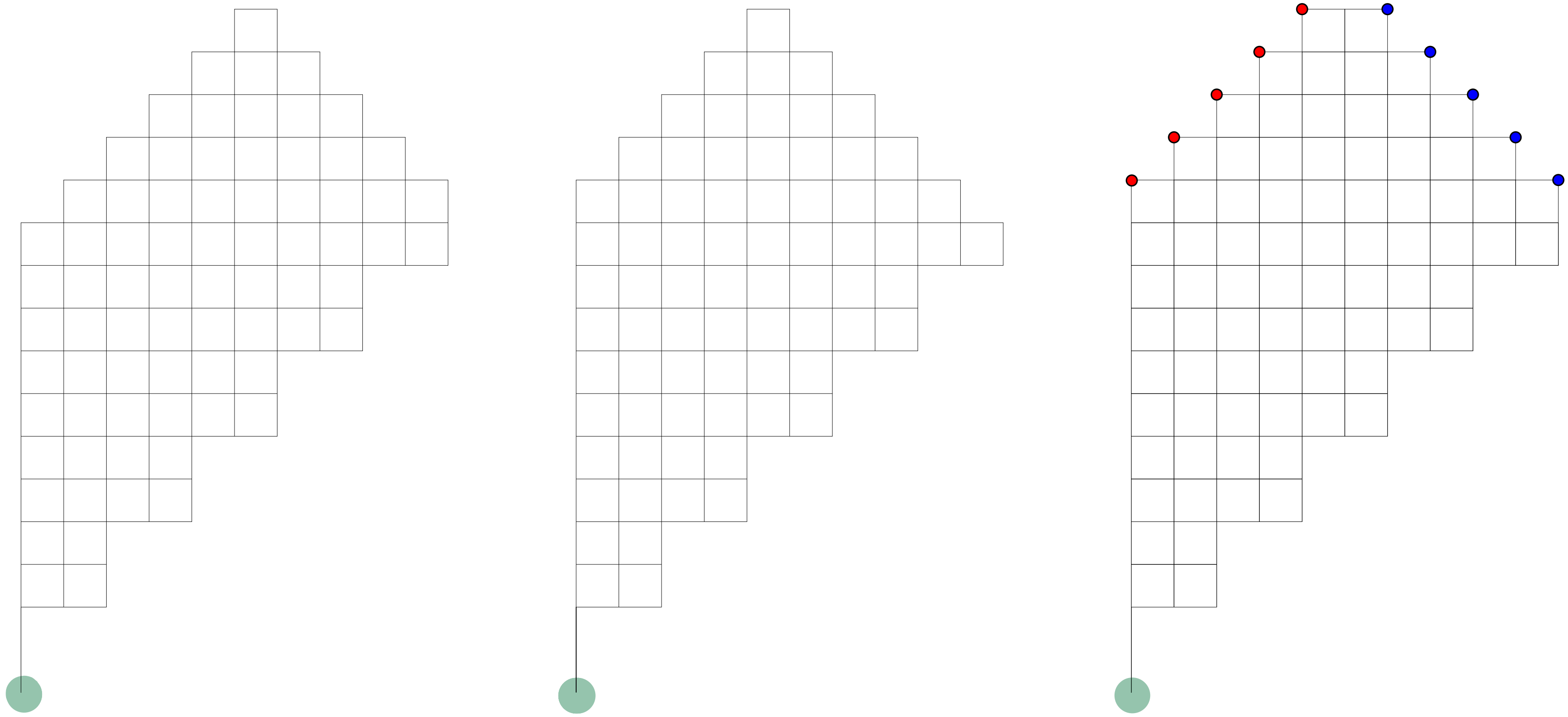
Dual graphs of $\mathcal{T}_n$ and $\mathcal{T}'_n$ of order 6.

Equivalent question

Find a bijection between sets of perfect matchings of these graphs.

* We use the same notations for the dual graphs: $\mathcal{T}_n$ and $\mathcal{T}'_n$.

Superposition of the graphs $\mathcal{T}_n$ and $\mathcal{T}'_n$

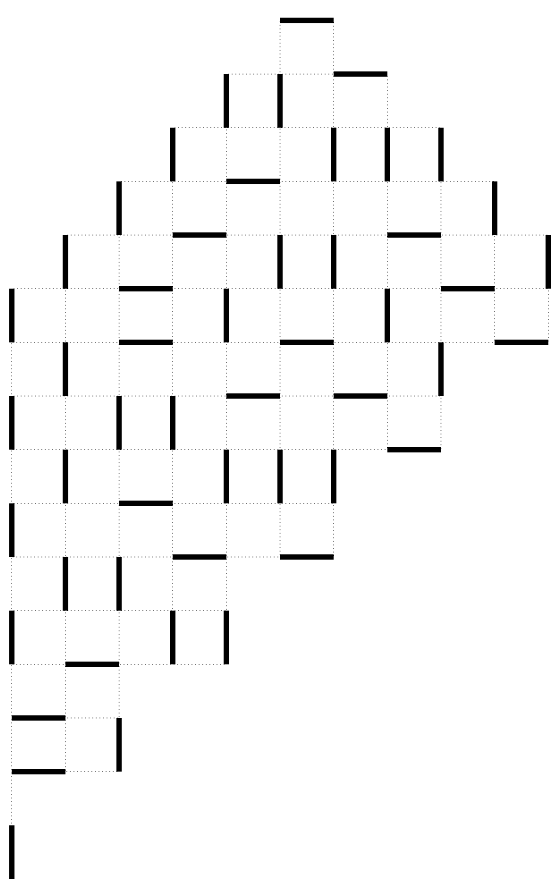


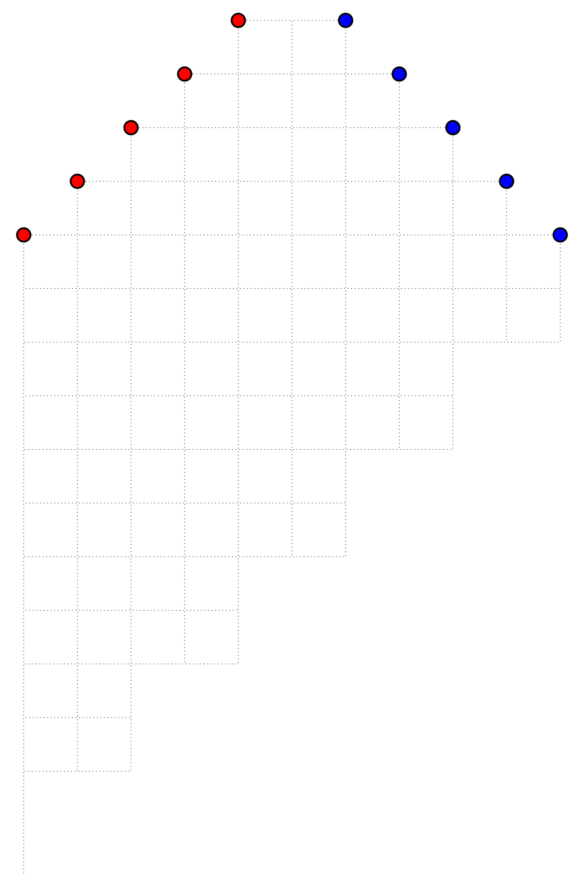
$\mathcal{T}_6$, $\mathcal{T}'_6$, and the their superposition

Gliding again

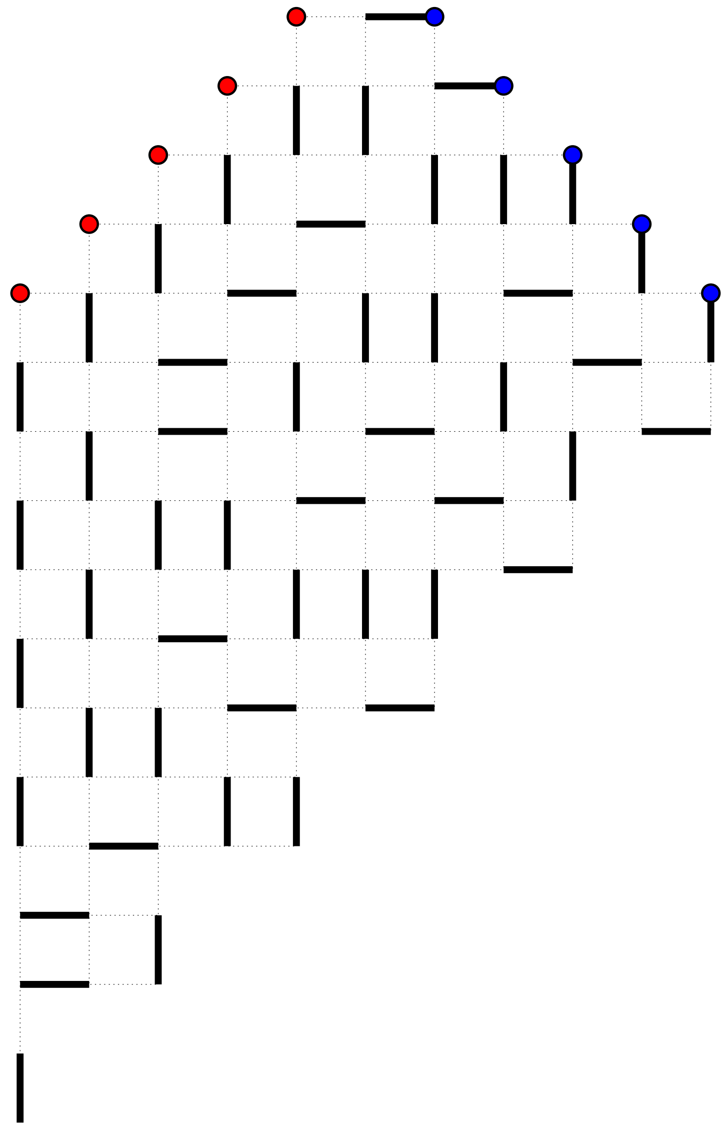
We consider a perfect matching of $\mathcal{T}_n$, and put the matching on the superposition graph. Then, we glide on vertices of $\mathcal{T}_n$ that are not in $\mathcal{T}'_n$.

(blue vertices)

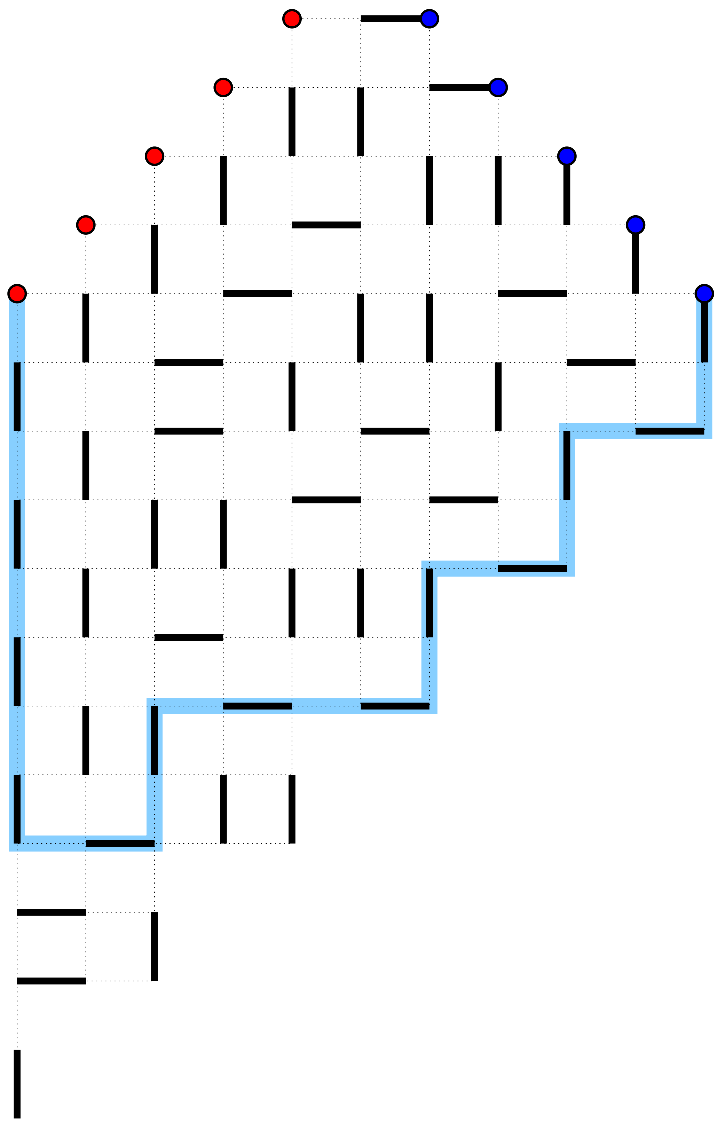




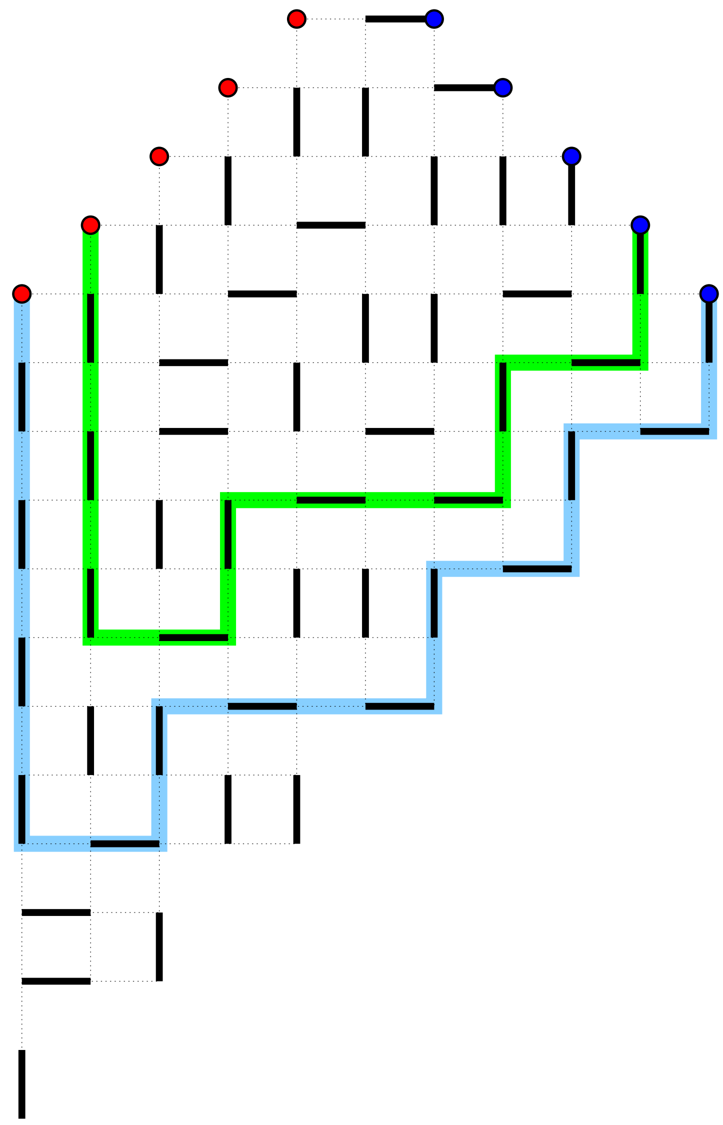
Gliding again



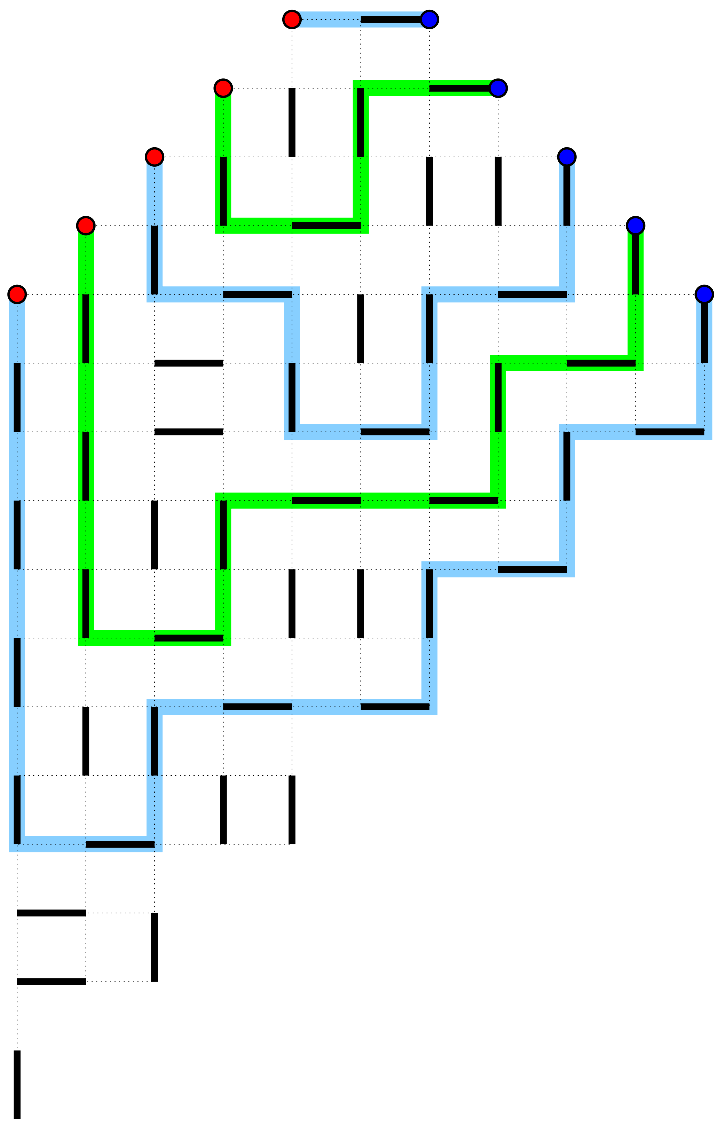
Gliding again



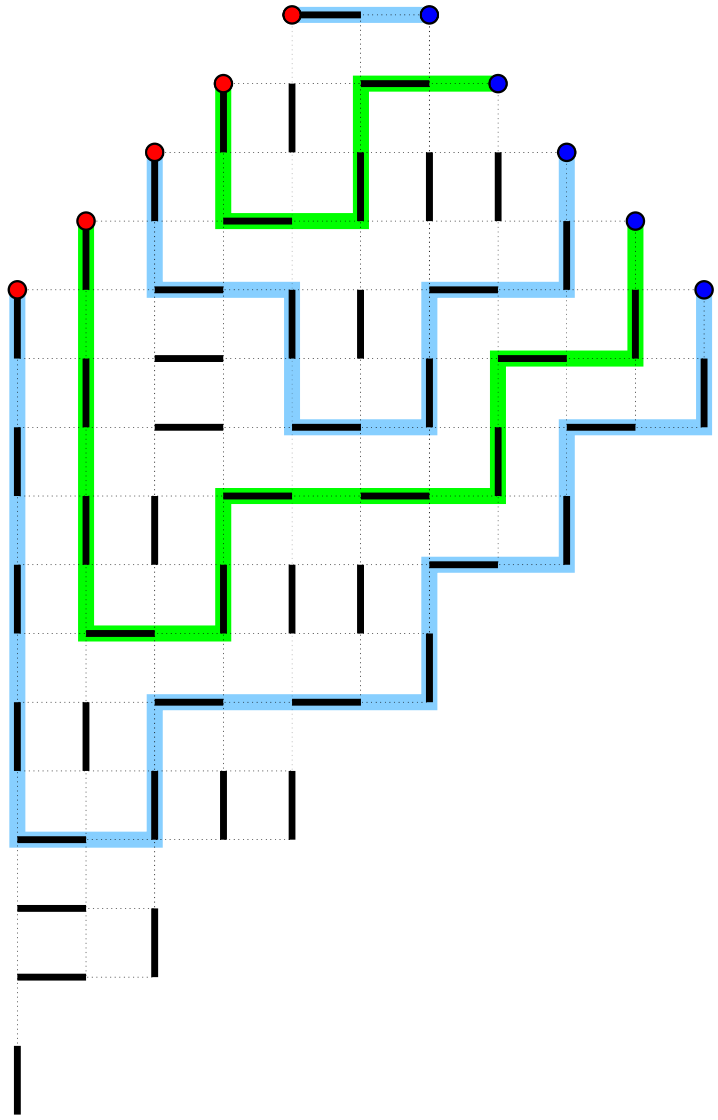
Keep gliding



At the end, we have...



Taking complement on colored paths



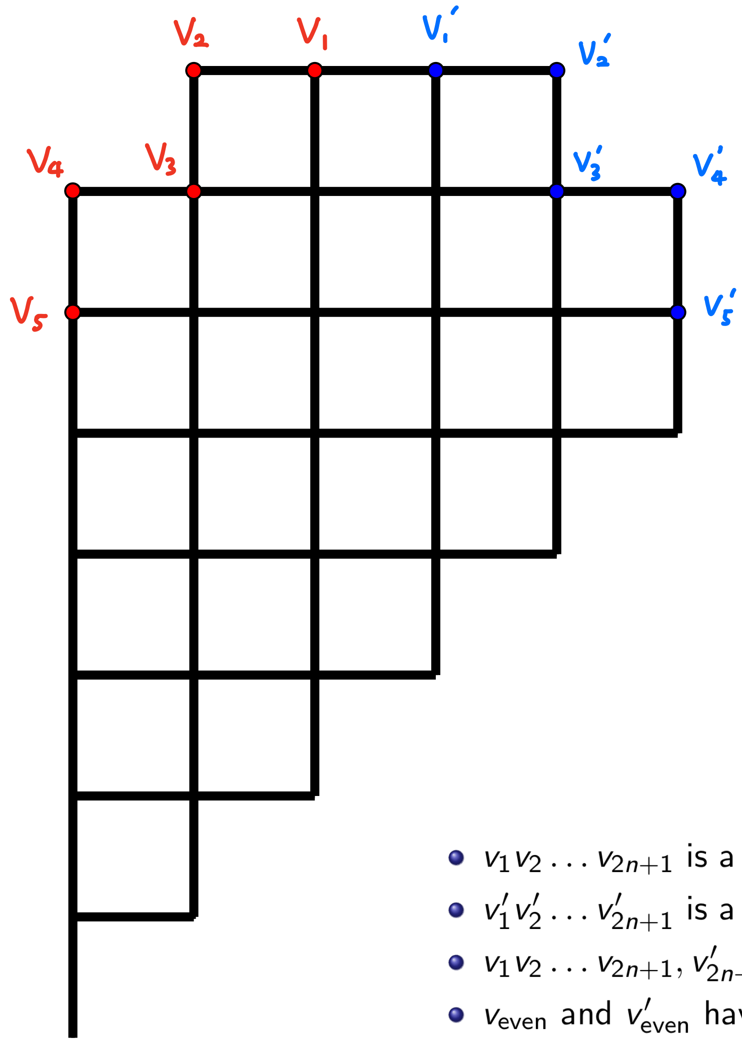
This is a perfect matching of T'_n.

Revealing the structure

Step 1: consider a generic plane graph G with $(4n+2)$ vertices $v_1, \dots, v_{2n+1}$ and $v'_1, \dots, v'_{2n+1}$ on the boundary of its unbounded face such that

- $v_1 v_2 \dots v_{2n+1}$ is a path in G,
- $v'_1 v'_2 \dots v'_{2n+1}$ is a path in G,
- $v_1 v_2 \dots v_{2n+1}, v'_{2n+1}, v'_{2n}, \dots, v'_1$ are in cyclic order,
- v_{even} and v'_{even} have degree two, and they belong to distinct bounded faces of G.

Revealing the structure

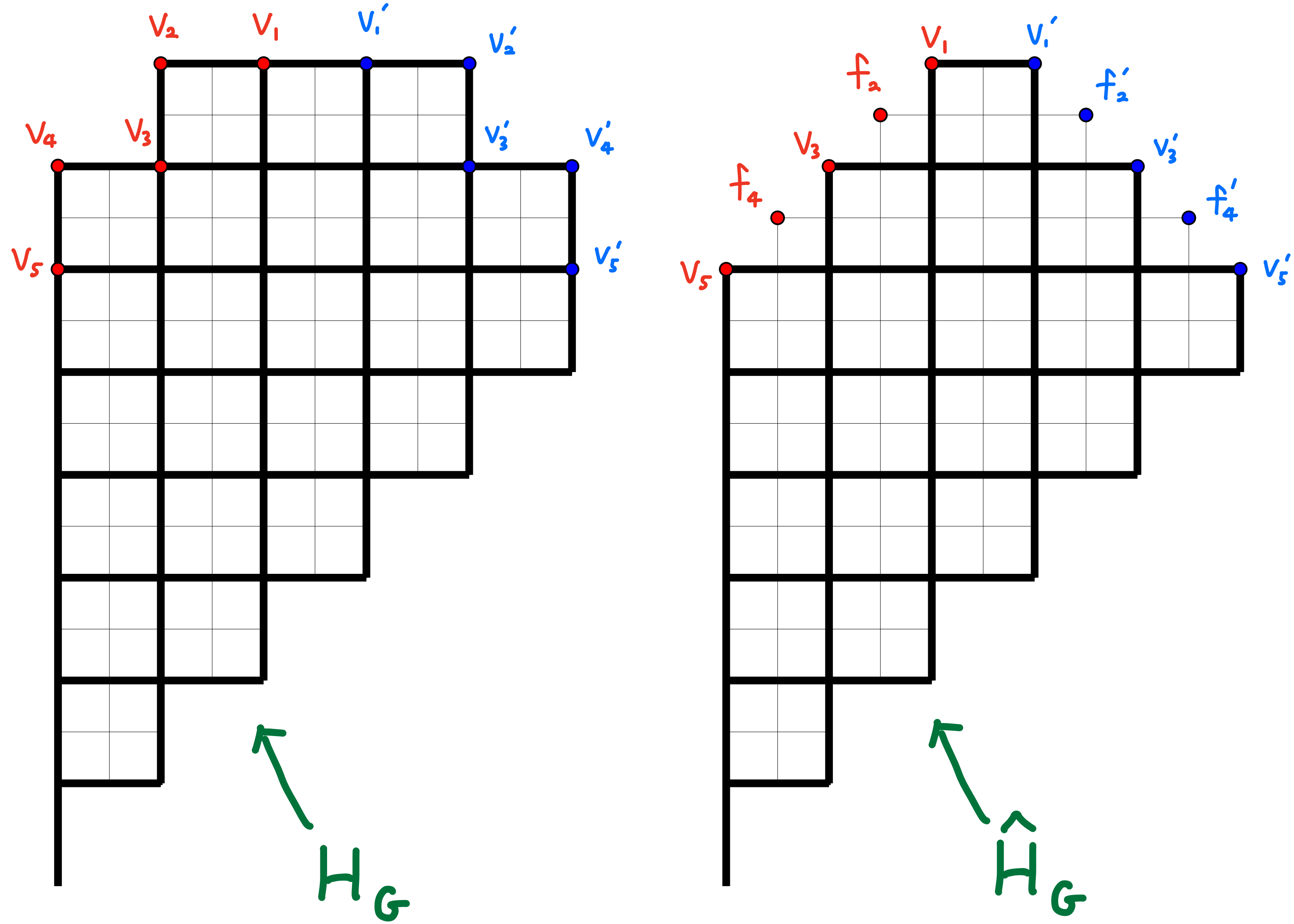


- $v_1 v_2 \dots v_{2n+1}$ is a path in G,
- $v'_1 v'_2 \dots v'_{2n+1}$ is a path in G,
- $v_1 v_2 \dots v_{2n+1}, v'_{2n+1}, v'_{2n}, \dots, v'_1$ are in cyclic order,
- v_{even} and v'_{even} have degree two, and they belong to distinct bounded faces of G.

Revealing the structure

Step 2: We take the dual refinement of G (we denote it by H_G). Then, we delete all v_{even} and v'_{even} and their nearest neighbors. Instead, we label face vertices of faces that contain v_{even} and v'_{even} by f_{even} and f'_{even}, respectively. The resulting graph is denoted by $\hat{H}_G$.

Revealing the structure



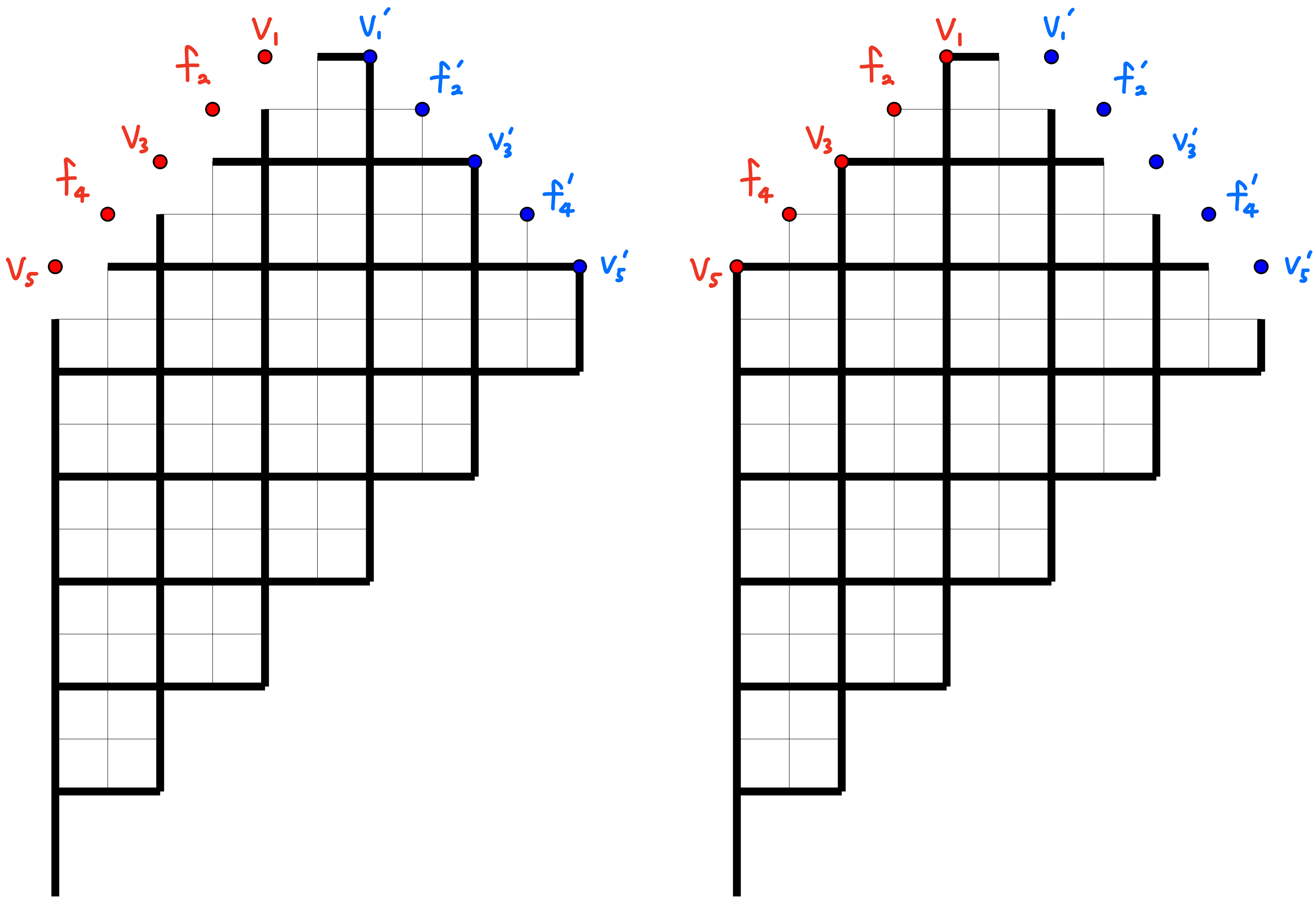
Revealing the structure

Step 3: Consider two graphs $\hat{H}_G \setminus \{v_1, f_2, \dots, f_{2n}, v_{2n+1}\}$ and $\hat{H}_G \setminus \{v'_1, f'_2, \dots, f'_{2n}, v'_{2n+1}\}$.

Theorem [B., Ciucu, 24+]

There is a bijection between perfect matchings of the graphs $\hat{H}_G \setminus \{v_1, f_2, \dots, f_{2n}, v_{2n+1}\}$ and $\hat{H}_G \setminus \{v'_1, f'_2, \dots, f'_{2n}, v'_{2n+1}\}$.

Revealing the structure



$\hat{H}_G \setminus \{v_1, f_2, \ldots, f_{2n}, v_{2n+1}\}$ $\hat{H}_G \setminus \{v_1', f_2', \ldots, f_{2n}', v_{2n+1}'\}$

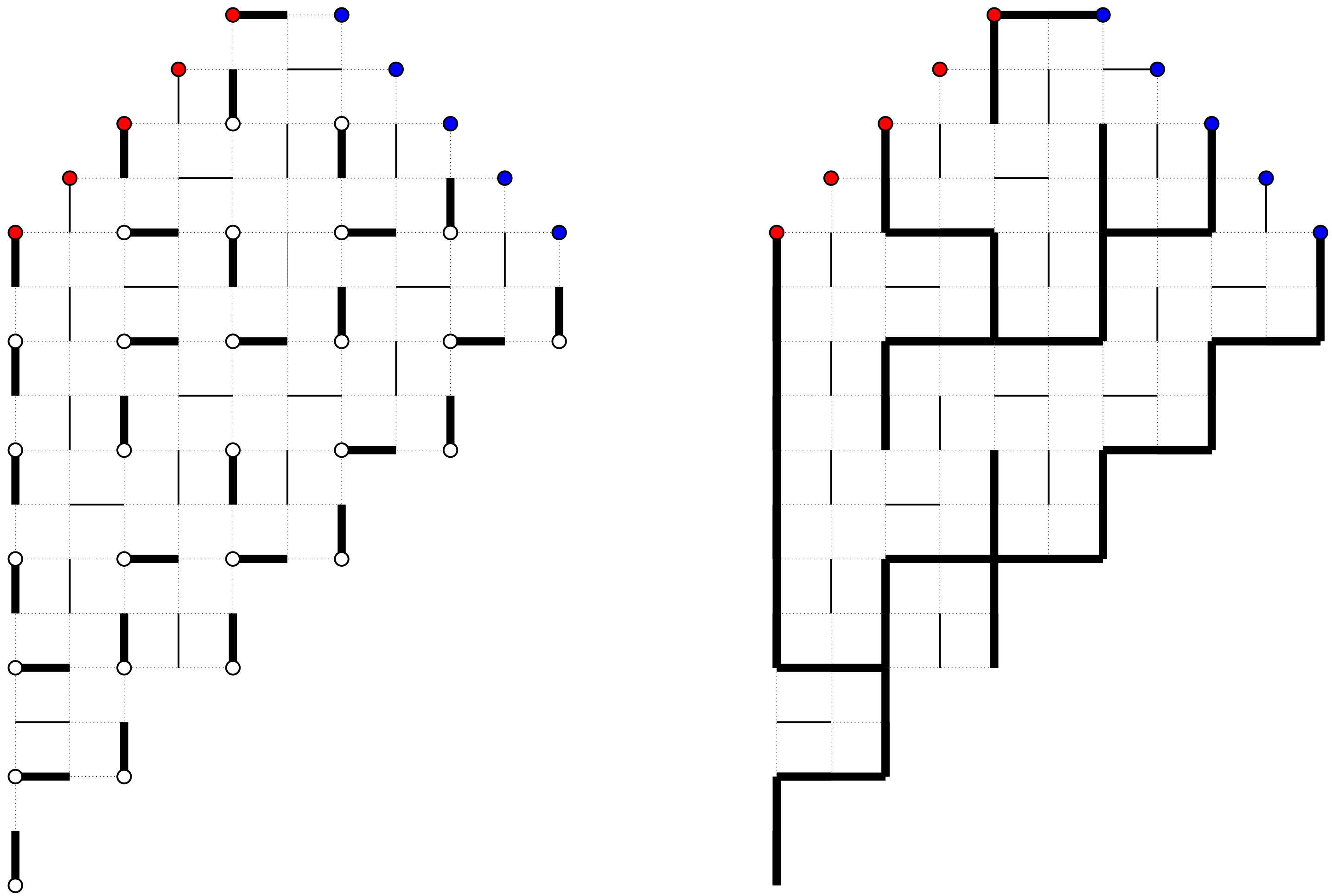
Perfect matchings and spanning trees

Based on Temperley's "Spanning tree-Perfect matching" bijection and a perfect matching bijection in Kenyon, Propp, and Wilson's paper, one can wonder if there is a corresponding "Spanning forest-Perfect matching" bijection.

Answer is No, but Yes if we only consider a set consisting of some *special* spanning forests.

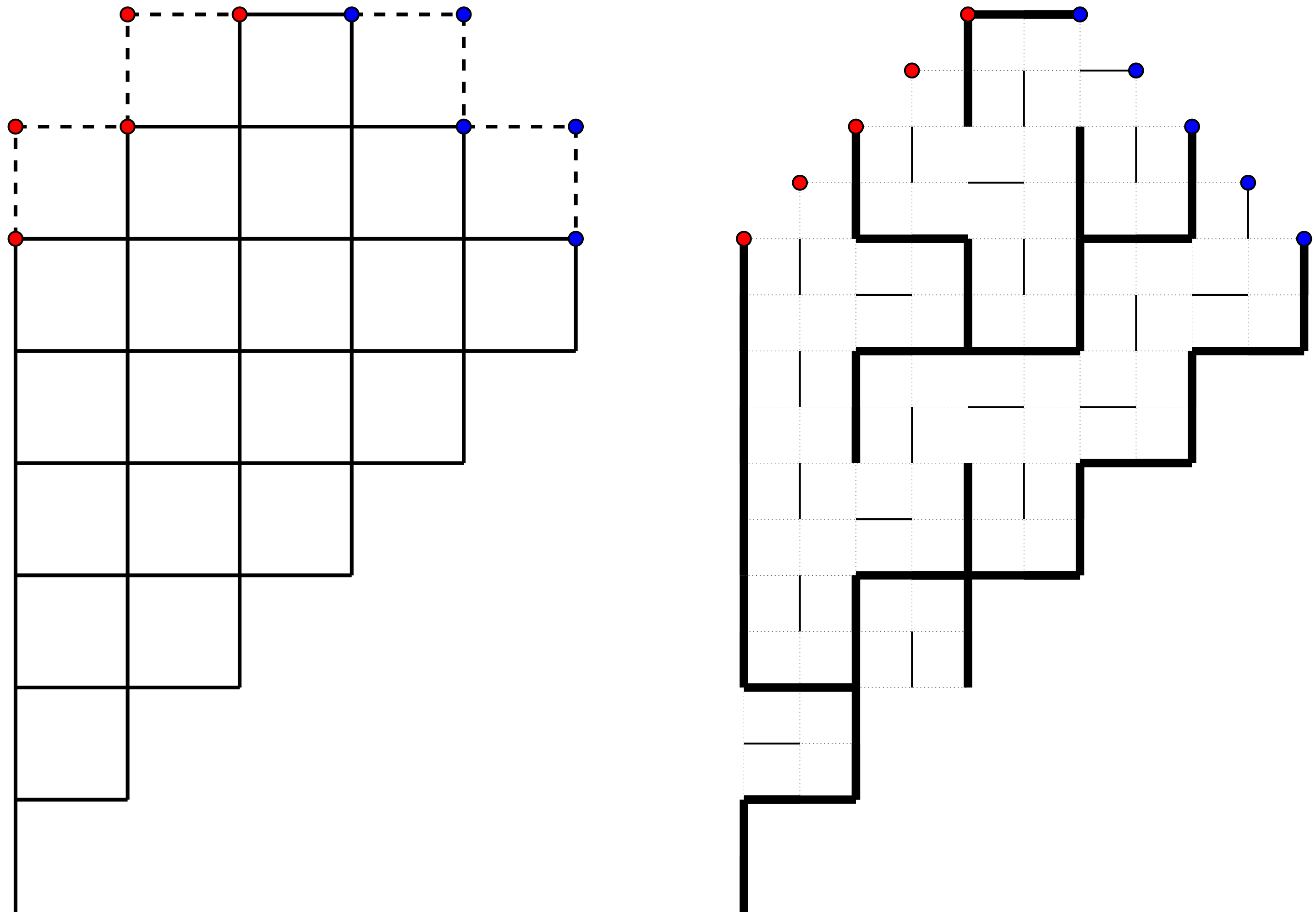
From perfect matchings to spanning forests

We use the same construction as Temperley: Given a perfect matching of $\hat{H}_G \setminus \{v'_1, f'_2, \dots, f'_{2n}, v'_{2n+1}\}$, we "double" edges on G.



From perfect matchings to spanning forests

What we get is a spanning forest of $G \setminus \{v_{\text{even}}, v'_{\text{even}}\}$.



From perfect matchings to spanning forests

What we got (in the previous slide) is not an arbitrary spanning forest. It has the following properties:

1. it consists of $n+1$ directed trees rooted at v'_{2i+1} for $i = 0, 1, \ldots, n$.
2. each tree rooted at v'_{2i+1} contains a vertex v_{2i+1} for $i = 0, 1, \ldots, n$.
3. two vertices f_{2i} and f'_{2i} (or equivalently v_{2i} and v'_{2i}) are not separated by spanning forests for $i = 1, \ldots, n$.

Theorem [B., Ciucu. 24+]

There is a bijection between perfect matchings of $\hat{H}_G \setminus \{v'_1, f'_2, \ldots, f'_{2n}, v'_{2n+1}\}$ and spanning forests of $G \setminus \{v_{\text{even}}, v'_{\text{even}}\}$ that satisfy the conditions above.

References

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Thank you!