Mallows Product Measure

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Overview

1. Stationary blocking measures of ASEP
   - One-species ASEP and stationary blocking measure
   - Infinite-species ASEP and Mallows measure
   - Mallows Product Measure
2. Exact formulas for several observables
   - The joint distribution of neighboring displacements
   - The (partial) joint distribution of arbitrary elements
   - Projection to the Bernoulli product measure
3. Application in multi-species interacting particle systems
   - ASEP with d second class particles
   - ASEP with one second class, one third class, $\cdots$, one $(d+1)$-st class particles
   - Projection to random walk on Hecke algebra

Definition

The (standard) asymmetric simple exclusion process is a collection of particles on $\mathbb{Z}$ which evolves in time:

- Asymmetric: Each particle jumps one step to the right with rate 1, and jumps one step to the left with rate q; $0 \leq q < 1$;
- Exclusion: Each particle can jump to the neighboring position only if the target position is vacant.

[Liggett, 1976]

1. Non-reversible: translation-invariant i.i.d. Bernoulli measures;
2. Reversible: Blocking measure (with probability 1 a random configuration consists of only holes to the left of a specific position, and consists of only particles to the right of another specific position.)
   - Product blocking measure μ_α^p; $\alpha \in \mathbb{R}_{>0}$: integer $i \in \mathbb{Z}$ is occupied by a particle with probability $\frac{1}{1+\alpha q^{i+1/2}}$ independently for all i;
   - Ergodic blocking measure $\mu_{(c)}$ (up to shift c): probability of a configuration is proportional to the $q^{\text{number of inversions}}$.
   - Ergodic decomposition[Borodin,2007; Balazs-Bowen, 2016; Betea-Bouttier, 2018]:

$$\mu_\alpha^p = \sum_{c\in\mathbb{Z}} \frac{\alpha^c q^{c^2/2}}{(q;q)_\infty \prod_{k=0}^{\infty}\left(1+\alpha q^{k+1/2}\right)\prod_{k=0}^{\infty}\left(1+\alpha^{-1}q^{k+1/2}\right)}\mu_{(c)}. \tag{1}$$

where the weights come from Jacobi triple product formula:

$$\sum_{c\in\mathbb{Z}} \frac{\alpha^c q^{c^2/2}}{(q;q)_\infty \prod_{k=0}^{\infty}\left(1+\alpha q^{k+1/2}\right)\prod_{k=0}^{\infty}\left(1+\alpha^{-1}q^{k+1/2}\right)} = 1. \tag{2}$$

Infinite-species ASEP on $\mathbb{Z}$

Definition

The Infinite-species asymmetric simple exclusion process is a collection of particles of various species (=classes, colors, types) on $\mathbb{Z}$ which evolve in time:

- infinite-species: configurations of the process are given by infinite permutations $\pi : \mathbb{Z} \to \mathbb{Z}$;
- A particle of a larger species interacts with a particle of a smaller species as a particle with a hole;
- Standard ASEP rules: neighboring particles rearrange themselves into increasing order at rate 1, and into decreasing order at rate q.

Projection:

- If we map all the species > 0 to the same species and call it a "particle"; and we map all the species ≤ 0 to the same type and call it a "hole", then we get the one-species ASEP from infinite-species ASEP.
- Similarly, one can also consider the multi-species ASEP model with finitely many species as the degeneration of the infinite-species ASEP.

Mallows measure on $\mathbb{Z}$

- Stationary reversible measures of infinite-species ASEP: detailed balance equation.
- The q-exchangeable probability measure on the infinite group of permutations of integers on $\mathbb{Z}$.
- The q-exchangeable probability measure $\mathcal{M}$ satisfies the following condition: for any $a > b$,
$$\mathcal{M}(...ab...) = q\mathcal{M}(...ba...). \tag{3}$$
- The ergodic q-exchangeable measures $\{\mathcal{M}_c\}_{c\in\mathbb{Z}}$ on permutations $\mathbb{Z} \to \mathbb{Z}$ were classified by [Gnedin-Olshanski, 2012]. They called them Mallows measure.
- Each q-exchangeable probability measure on the infinite group of permutations of integers is a unique convex mixture of the Mallows measures $\mathcal{M}_c$ over $c \in \mathbb{Z}$. [Gnedin-Olshanski, 2012]

The joint distribution of displacements

Let ω be a random permutation $\mathbb{Z} \to \mathbb{Z}$, distributed according to Mallows measure $\mathcal{M}_c$, $c \in \mathbb{Z}$. We denote $(q;q)_k = \prod_{i=1}^{k}(1-q^i)$.

Theorem (Gnedin-Olshanski, 2012)

For integers $d_1 \leq \cdots \leq d_k$, the joint distribution of displacements $D_i = \omega(i) - i$ is given by

$$\mathbb{P}\left(D_1 = d_1, \dots, D_k = d_k\right) = (1-q)^k q^{-k(k+1)/2} (q;q)_\infty$$
$$\times \prod_{m=2}^{k} (q;q)_{d_m - d_{m-1}} \sum \frac{q^{\sum_{1 \leq i \leq j \leq k} (b_i+1)\left(a_j+1\right)}}{(q;q)_{b_1} \dots (q;q)_{b_k} (q;q)_{a_1} \dots (q;q)_{a_k}}, \tag{4}$$

where the summation is over all nonnegative integers $a_1, b_1, \dots, a_k, b_k$ which satisfy the constraints

$$(b_1 + \cdots + b_m) - (a_m + \cdots + a_k) = d_m - c, \quad m = 1, \dots, k. \tag{5}$$

- It is very difficult to write a reasonable formula for the joint dirtribution of displacements $(D_{i_1}, D_{i_2}, \cdots, D_{i_k})$ with arbitrary indices $i_1 < i_2 < \cdots < i_k$.

- At the level of one-species ASEP, the product blocking measures are much simpler than the ergodic ones.
- The simplest infinite-ASEP stationary blocking measures are not ergodic measures, but rather certain mixtures of them.
- Bufetov-Chen Mallows Product Measure:

$$\mathcal{M}_\alpha^p := \sum_{c\in\mathbb{Z}} \frac{\alpha^c q^{c^2/2}}{(q;q)_\infty \prod_{k=0}^\infty (1+\alpha q^{k+1/2}) \prod_{k=0}^\infty (1+\alpha^{-1} q^{k+1/2})} \mathcal{M}_c. \tag{6}$$

- As an appetizer, we compute the distribution of $\omega(0)$ in the Mallows product measure.

$$\mathbb{P}(\omega(0)=x) = Z \sum_{c\in\mathbb{Z}} \alpha^c q^{c^2/2} \sum_{\{r,\ell\geq 0: r-\ell=x-c\}} \frac{q^{r\ell+r+\ell}}{(q;q)_\ell (q;q)_r}, \tag{7}$$

where $Z = (1-q)\prod_{k=0}^\infty \left(1+\alpha q^{k+1/2}\right)^{-1} \prod_{k=0}^\infty \left(1+\alpha^{-1} q^{k+1/2}\right)^{-1}$.

- By using the identity of Euler and the finite q-binomial theorem, we have

$$\mathbb{P}(\omega(0)=x) = \frac{(1-q)\alpha q^{x-1/2}}{(1+\alpha q^{x-1/2})(1+\alpha q^{x+1/2})}. \tag{8}$$

Let ω be the random permutation of $\mathbb{Z}$ distributed according to the Mallows product measure $\mathcal{M}^{p}_{\alpha}$.

Theorem (Bufetov-Chen, 2024)

Let $x_1 \le x_2 \le \cdots \le x_k$ be integers, the joint distribution of k neighboring displacements $D_j = \omega(j) - j$ for $j = 1, 2, \cdots, k$ is given by

$$\mathbb{P}\left(D_1 = x_1, D_2 = x_2, \cdots, D_k = x_k\right) = \frac{(1-q)^k \alpha^k q^{\sum_{j=1}^{k} x_j - \frac{k^2}{2}}}{\prod_{j=1}^{k}(1+\alpha q^{x_j+2j-k-\frac{3}{2}})(1+\alpha q^{x_j+2j-k-\frac{1}{2}})}. \tag{9}$$

- q-exchangeability: the constraint $x_1 \le x_2 \le \cdots \le x_k$ can be removed.
- Translation-invariance: the joint distribution of the neighboring displacements $D_i = \omega(i) - i$ for $i = 1, 2, \cdots, k$ does not change if we simultaneously shift all indices $1, 2, \cdots, k$ by a constant.
- Inversion-invariance: the Mallows product measure is invariant under the inversion map: $\omega \to \omega^{-1}$.

The (partial) joint distribution of arbitrary elements

Theorem (Bufetov-Chen, 2024)

Let $x_1, x_2, \cdots, x_k$ be k integers and $x_1 > x_2 > \cdots > x_k$. For any k integers $i_1 < i_2 < \cdots < i_k$, one has

$$\mathbb{P}\left(\omega(i_1) = x_1, \omega(i_2) = x_2, \cdots, \omega(i_k) = x_k\right) = \prod_{j=1}^{k} \frac{(1-q)\alpha q^{x_j - i_j - \frac{1}{2}}}{(1+\alpha q^{x_j - i_j - \frac{1}{2}})(1+\alpha q^{x_j - i_j + \frac{1}{2}})}. \tag{10}$$

- Shift-invariance symmetry of the stochastic colored six-vertex model [Borodin-Gorin-Wheeler, 2019; Galashin, 2021]
- Limit transition from stochastic six-vertex model to the ASEP [Boroin-Corwin-Gorin, 2014; Aggarwal, 2017]

Theorem (Bufetov-Chen, 2024)

Let $i_1 < i_2 < \cdots < i_k$, $x_1 > x_2 > \cdots > x_k$, $j_1 < j_2 < \cdots < j_k$, $y_1 > y_2 > \cdots > y_k$ be integers such that $x_a - i_a = y_a - j_a$, for any $a = 1, 2, \ldots, k$, and $k \in \mathbb{N}$. We have

$$\begin{aligned}\mathbb{P}\left(\omega(i_1) = x_1, \omega(i_2) = x_2, \cdots, \omega(i_k) = x_k\right)& \\ = \mathbb{P}\left(\omega(j_1) = y_1, \omega(j_2) = y_2, \cdots, \omega(j_k) = y_k\right).\end{aligned} \tag{11}$$

Projection to the Bernoulli product measure

Theorem (Bufetov-Chen, 2024)

Let $i_1 < i_2 < \cdots < i_k$, $x_1 \geq x_2 \geq \cdots \geq x_k$, $j_1 < j_2 < \cdots < j_k$, $y_1 \geq y_2 \geq \cdots \geq y_k$ be integers such that $x_a - i_a = y_a - j_a$, for any $a = 1, 2, \ldots, k$, and $k \in \mathbb{N}$. We have

$$\mathbb{P}\left(\omega(i_1) \leq x_1, \omega(i_2) \leq x_2, \cdots, \omega(i_k) \leq x_k\right) = \mathbb{P}\left(\omega(j_1) \leq y_1, \omega(j_2) \leq y_2, \cdots, \omega(j_k) \leq y_k\right). \quad (12)$$

Theorem (Bufetov-Chen, 2024)

Let $x_1 \geq x_2 \geq \cdots \geq x_k$, $i_1 < i_2 < \cdots < i_k$ be integers. One has

$$\mathbb{P}\left(\omega(i_1) \leq x_1, \omega(i_2) \leq x_2, \cdots, \omega(i_k) \leq x_k\right) = \prod_{j=1}^{k} \frac{1}{1 + \alpha q^{x_j - i_j + \frac{1}{2}}}. \quad (13)$$

- For $x_1 = x_2 = \cdots = x_k = 0$, one has

$$\mathbb{P}\left(\omega(i_1) \leq 0, \omega(i_2) \leq 0, \cdots, \omega(i_k) \leq 0\right) = \prod_{j=1}^{k} \frac{1}{1 + \alpha q^{-i_j + \frac{1}{2}}} \quad (14)$$

  which implies that the one-species projection of the Mallows product measure is the Bernoulli product measure.

ASEP with d second class particles

Let ϕ be of the following projection:

$$\phi(i) = \begin{cases} 0, & i \leq 0 \\ 1, & i \in \{1, 2, \cdots, d\} \\ 2, & i > d \end{cases} \tag{15}$$

Then $\phi(\omega(\cdot))$, where ω is distributed according to the Mallows product measure, coincides with the product blocking measure of ASEP with d second class particles. We denote the position of the i-th second class particle by s_i, $i = 1, 2, \cdots, d$. Namely, we have $\phi(\omega(s_i)) = 1$ for $i = 1, 2, \cdots, d$, $s_1 < s_2 < \cdots < s_d$.

Theorem ($\alpha = q^{\frac{1}{2}-c}$: Adams-Balázs-Jay, 2023.)

Let $x_1 < x_2 < \cdots < x_d$ be arbitrary integers. One has

$$\mathbb{P}(s_1 = x_1, s_2 = x_2, \ldots, s_d = x_d) = \frac{\alpha^d q^{\sum_{i=1}^d x_i - \frac{d(2d+1)}{2}} \prod_{i=1}^d (1 - q^i)}{\prod_{j=1}^d (1 + \alpha q^{x_j + j - d - \frac{3}{2}})(1 + \alpha q^{x_j + j - d - \frac{1}{2}})}. \tag{16}$$

ASEP with one second class, one third class, $\cdots$, one $(d+1)$-st class particles

Let ϕ be of the following projection:

$$\phi(i)=\begin{cases}0, & i\leq 0\\ i, & i\in\{1,2,\cdots,d\}\\ d+1, & i>d\end{cases} \tag{17}$$

This projection corresponds to the product blocking measure of ASEP with one second class, one third class, $\cdots$, one $(d+1)$-st class particles. We denote the position of the i-th class particle by s_i, $i=2,3,\cdots,d+1$.

Theorem

For any distinct fixed $x_1,x_2,\cdots,x_d\in\mathbb{Z}$, the probability of the second class, the third class, $\cdots$, the $(d+1)$-st class particles staying at positions $x_1,x_2,\cdots,x_d$ is given by the following formula:

$$\begin{aligned}&\mathbb{P}\left(s_2=x_1,s_3=x_2,\cdots,s_{d+1}=x_d\right)\\ &\quad=q^{inv(\sigma)}\frac{(1-q)^d\alpha^d q^{\sum_{i=1}^d x_i-\frac{d(2d+1)}{2}}}{\prod_{j=1}^d(1+\alpha q^{x_{\sigma(d+1-j)}+j-d-\frac{3}{2}})(1+\alpha q^{x_{\sigma(d+1-j)}+j-d-\frac{1}{2}})},\end{aligned} \tag{18}$$

where σ is the permutation such that $x_{\sigma(d)}<x_{\sigma(d-1)}<\cdots<x_{\sigma(1)}$.

Projection to random walk on Hecke algebra

- A class of multi-species interacting particle systems can be realized as random walks on Hecke algebras. [Bufetov, 2020]
- Stochastic six-vertex model, ASEP(q,M), q-TAZRP, general M exclusion asymmetric process.
- Mallows (product) measure is a reversible stationary measure for these systems, and formulas above can be used to find properties of second, third, $\cdots$, class particles in these systems.

Thank You for Your Attention!