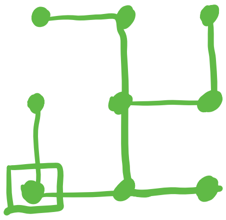
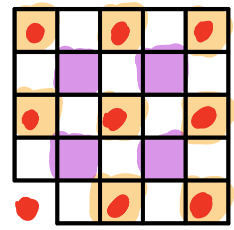
Lecture 2

Temperley bijection



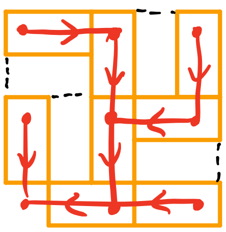
root

Spanning tree T on $\mathbb{Z}^2$



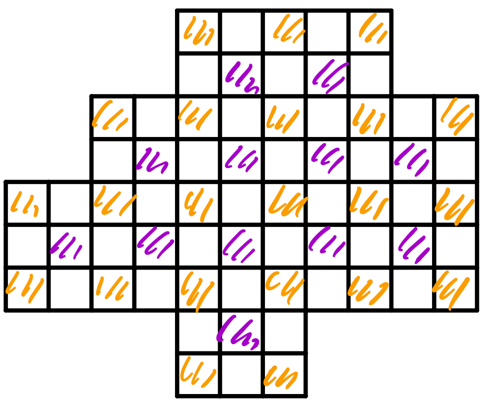
Two types of black faces B_0 and B_1

(vertices of $T \leftrightarrow B_0$)



Bijection between spanning trees and domino tilings

Def: Temperleyan domain:

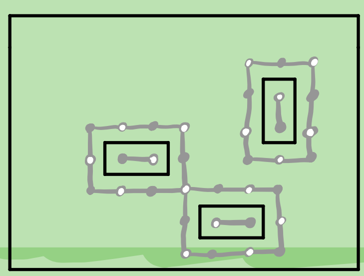


All corner squares are of type B_0.

Rmk: To get a tileable domain remove one B_0 square from the boundary.

Local statistics

Thm: [Kenyon '1997]

$\mathbb{P}\Big[$  $\Big]$

$$= \frac{\left|\det K_{(W \setminus \{w_j\}) \times (B \setminus \{b_j\})}\right|}{|\det K|}$$

Rmk:

$$\mathbb{P}(wb) = |K^{-1}(b,w)|$$

Fact:

big submatrix

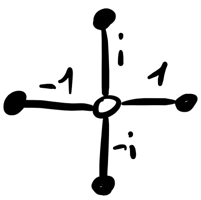
small submatrix

$$\frac{\left|\det K_{(W\setminus\{w_j\})\times(B\setminus\{b_j\})}\right|}{|\det K|} = \left|\det K^{-1}_{\{b_j\}\times\{w_j\}}\right|$$

- All local statistics for the uniform measure on dimer configurations on a planar graph can be computed using the inverse of Kasteleyn matrix.

K^{-1} as a $\bar{\partial}$-operator

Consider Kasteleyn signs (introduced by Kenyon):



Note that $K^{-1} \cdot K = Id$

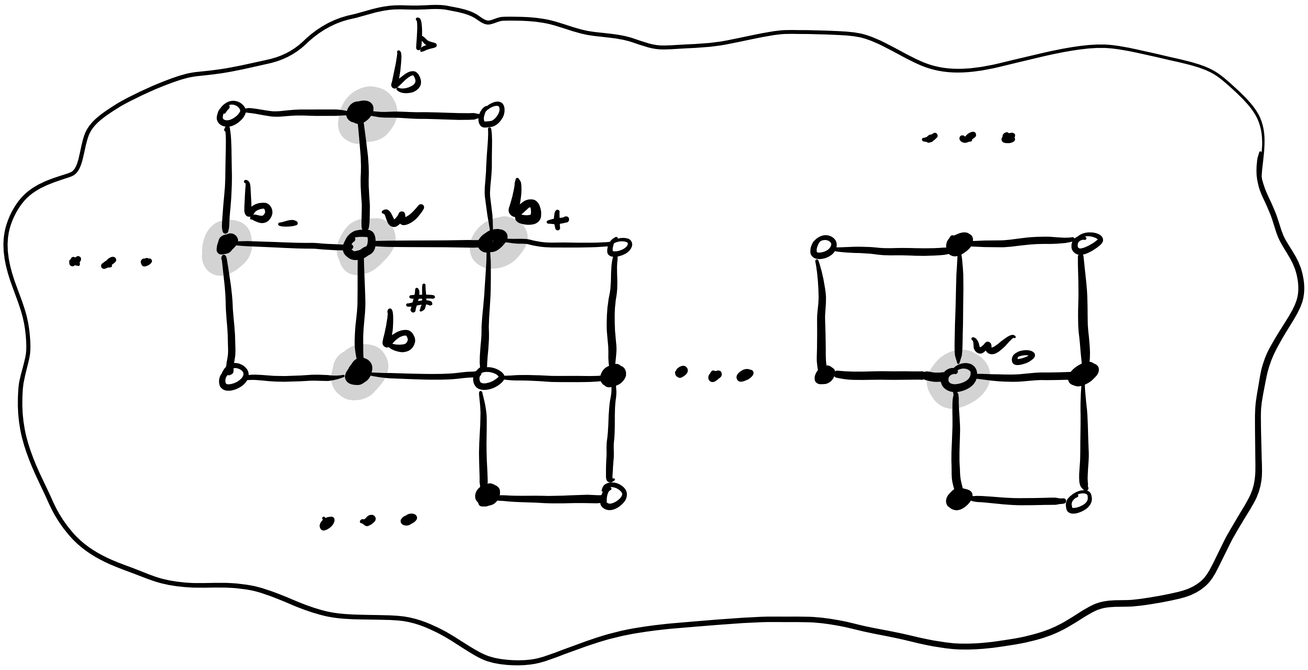
$$\Rightarrow \sum_{B} K^{-1}(b, w_0)\, K(w, b) = Id(w_0, w)$$

Since K is signed adjacency matrix, we have

$$K^{-1}(b_+, w_0) - K^{-1}(b_-, w_0) + i K^{-1}(b^\flat, w_0) - i K^{-1}(b^\sharp, w_0)$$

$$= \delta_{w = w_0},$$

where



Define

$$\left[F_{w_0}(b) = \right] F(b) := K^{-1}(b, w_0).$$

Then

$$F(b_+) - F(b_-) = i\,\big(F(b^\#) - F(b^\flat)\big)$$

$$= \frac{F(b^\flat) - F(b^\#)}{i}$$

$\boxed{\text{if } w \neq w_0}$

discrete Cauchy-Riemann equation

F is discrete holomorphic at all $w \neq w_0$

Divide black squares into two classes

B_0 and B_1

$F|_{B_0}$ satisfies the following

$$4F(b) = F(b+2) + F(b-2) + F(b+2i) + F(b-2i)$$

for all $b \in B_0$ such that $b \neq w_0$.

$F|_{B_0}$ is discrete harmonic at all $b \neq w_0$

- At w_0:

$$F(w_0+1) - F(w_0-1) + iF(w_0+i) - iF(w_0-i) = 1$$

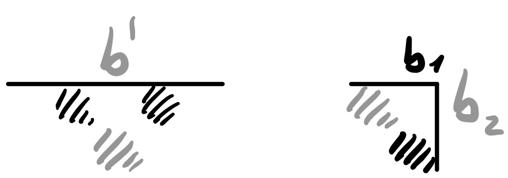
$$\Sigma\, F(b^*) - F(w_0+1) = 1$$

$$\Sigma\, F(b^*) - F(w_0-1) = -1$$

In other words: Away from the boundary

$$\begin{cases} [\bar\partial F](w) = \delta_{w=w_0} \\ [\Delta F](b) = \delta_{b=w_0+1} - \delta_{b=w_0-1} \end{cases}$$

- Near the boundary:

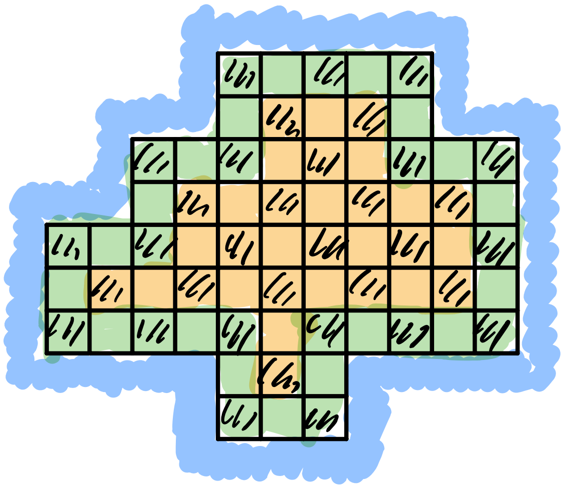


$$F(b') := 0$$

$$\begin{cases} F(b_1) := 0 \\ F(b_2) := 0 \end{cases} \Rightarrow$$

$[\bar\partial F](w) = \delta_{w=w_0}$ $\forall\, w \in \Omega$, and

$$[\Delta F](b) = \delta_{b=w_0+1} - \delta_{b=w_0-1} \quad \forall\ b \in \Omega \setminus \partial_{int}\Omega$$



Notation:

$\partial_{int}\Omega$

$\partial\Omega$

$Int\,\Omega$

$$\Omega = Int\,\Omega \cup \partial_{int}\Omega$$
$$\overline{\Omega} = \Omega \cup \partial\Omega$$

Claim: $F|_{B_0} \in \mathbb{R}$ and $F|_{B_1} \in i\mathbb{R}$

Then $K^{-1}(\cdot, w_0)$ is discrete holomorphic, with a singularity at w_0. Moreover, $K^{-1}(\cdot, w_0)|_{B_{0,1}}$ is discrete harmonic at all black $b \in Int\,\Omega \setminus \{w_0 \pm 1, w_0 \pm i\}$. And

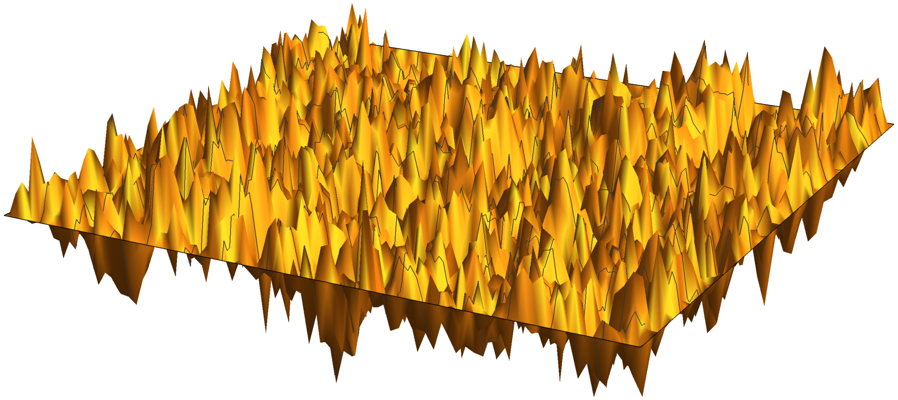
$$\begin{cases} K^{-1}(\cdot, w_0)|_{B_0} = \operatorname{Re} K^{-1}(\cdot, w_0) \\ K^{-1}(\cdot, w_0)|_{B_1} = i \operatorname{Im} K^{-1}(\cdot, w_0) \end{cases}$$

Claim: On Temperleyan domains $\operatorname{Re} K^{-1}(\cdot, w_0)$ has zero Dirichlet boundary conditions.

Gaussian Free Field

The Gaussian Free Field is not a random function, but a random distribution.

[1d analog: Brownian Bridge]



The Gaussian free field Φ on $\mathcal{D}$ is the random distribution such that pairings with test functions $\int_{\mathcal{D}} f\Phi$ are jointly Gaussian with covariance

$$\operatorname{Cov}\left(\int_{\mathcal{D}} f_1\Phi, \int_{\mathcal{D}} f_2\Phi\right) = \int_{\mathcal{D}\times\mathcal{D}} f_1(z)G(z,w)f_2(w).$$

Height fluctuations converge to the GFF

① $w_0 \in W_0$, $b \in B_0$

$G(z, w)$, $z = x+iy$, $w = u+iv$

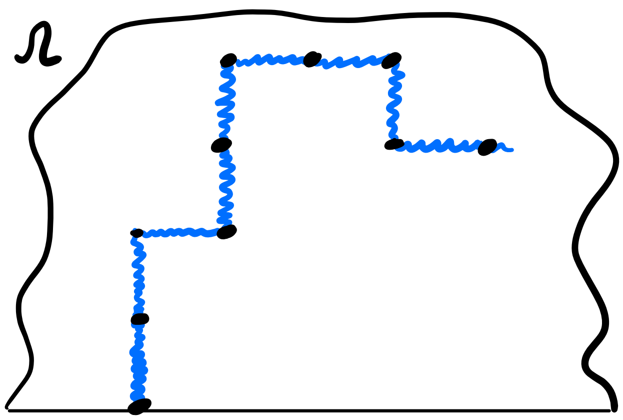
$$\frac{1}{2\delta} K^{-1}(b, w_0) \underset{\delta \searrow 0}{\rightrightarrows} -\frac{\partial}{\partial u} G(b, w_0)$$

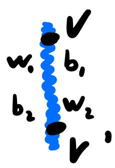
uniformly on compacts
(holds near straigh part of $\partial\Omega$)

More precisely,

$$\frac{1}{2\delta}\Big(K^{-1}(b, w_0) - \underbrace{F(b, w_0)}_{=\begin{cases} \mathrm{Re}\,\frac{\delta}{\pi(b-w_0)}, & b \in B_0 \\ i\,\mathrm{Im}\,\frac{\delta}{\pi(b-w_0)}, & b \in B_1 \end{cases}}\Big) \underset{\delta \searrow 0}{\rightrightarrows} -\frac{\partial}{\partial u}\Big(G(b, w_0) - \frac{1}{2\pi}\log|b-w_0|\Big)$$

② $$\mathbb{E}[H(v) - H(v')) = +3 \cdot \mathbb{P}[w_1 b_1] - 1 \cdot (1 - \mathbb{P}[w_1 b_1])$$
$$-3 \cdot \mathbb{P}[w_2 b_2] + 1 \cdot (1 - \mathbb{P}[w_2 b_2])$$
$$= 4(\mathbb{P}[w_1 b_1] - \mathbb{P}[w_2 b_2])$$





Rmk: All moments of the height function can be written in terms of K^{-1}.

Thm: (Kenyon '00)

On Temperleyan domains height fluctuations converge to the GFF as $\delta \searrow 0$.