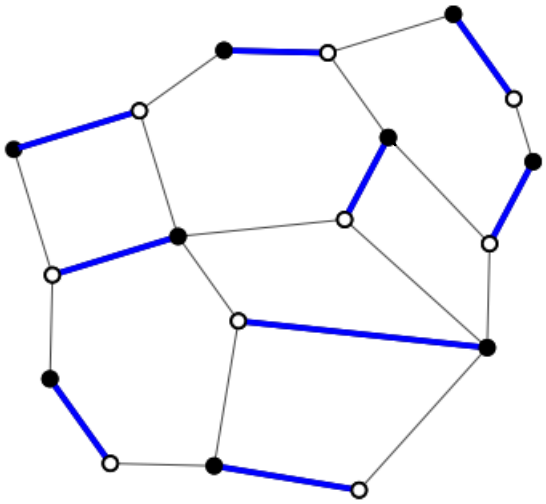
Lecture 3

(Centers of) circle patterns or t-embeddings.

Reminder:



Weight Function

$$\nu : E \to \mathbb{R}_{>0}$$

Probability measure on dimer configurations:

$$\mathbb{P}[m] = \frac{1}{Z} \underbrace{\prod_{e \in m} \nu(e)}_{=: \nu(m)},$$

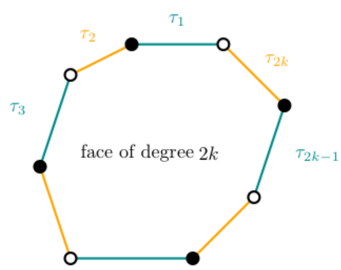
with

$$Z = \sum_{m \in \mathcal{M}} \nu(m).$$

Kasteleyn matrix

Complex Kasteleyn signs:
$\tau_i \in \mathbb{C}$, $|\tau_i| = 1$,

$$\frac{\tau_1}{\tau_2} \cdot \frac{\tau_3}{\tau_4} \cdot \ldots \cdot \frac{\tau_{2k-1}}{\tau_{2k}} = (-1)^{(k+1)}$$

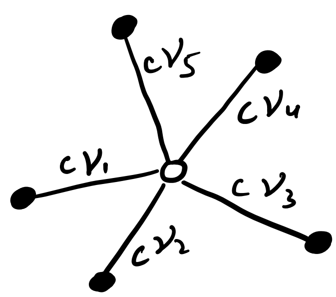


$K(w,b) = \nu_{wb} \tau_{wb}$ if $w \sim b$

$K(w,b) = 0$ otherwise.

Thm: $Z = |\det K|$

Gauge equivalence:



$\nu_1(wb) = F(b)\, \nu_2(wb)\, G(w)$

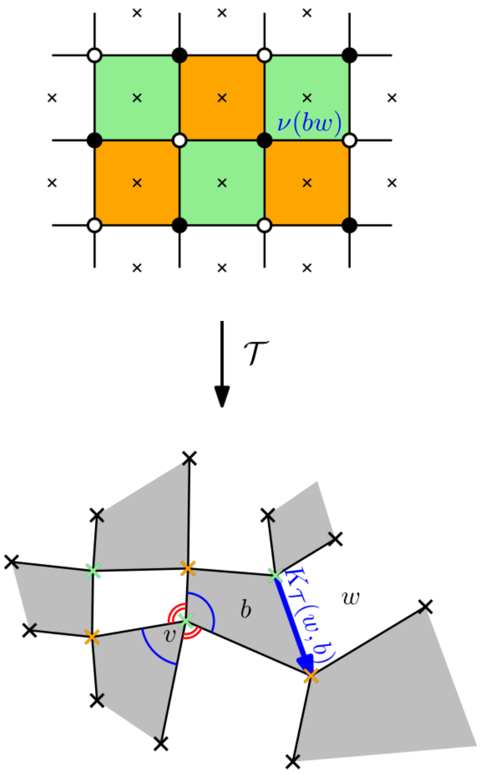
then $\nu_1 \underset{\text{gauge}}{\sim} \nu_2$

(F, G) – gauge functions

Rmk: Gauge equivalent weight functs define the same probability measure.

(Centers of) circle patterns or t-embeddings.

Definition: t-embedding



[Chelkak, Laslier, R.]

$\mathcal{T}$ is embedding of $\mathcal{G}^*$ such that

1) **lengths** are gauge equivalent to (given) dimer weights
2) **angles** at (inner) vertices are balanced:

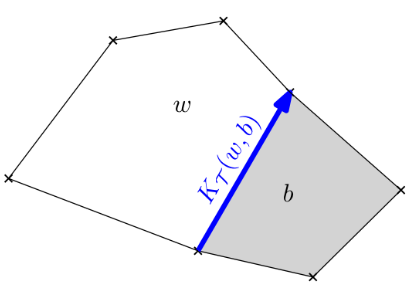
$$\sum_{f \text{ white}} \theta(f,v) = \sum_{f \text{ black}} \theta(f,v) = \pi.$$

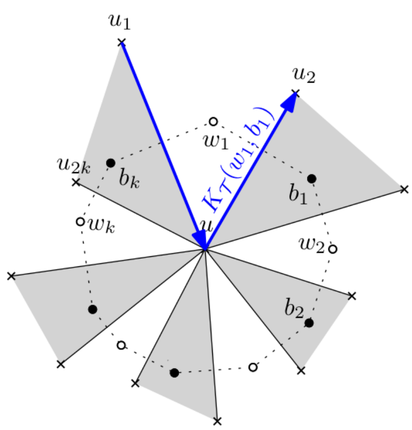
Rmk: (2) $\implies$ Kasteleyn sign condition.

$K_{\mathcal{T}}$ is a Kasteleyn matrix.

Kasteleyn weights

$$\mathcal{T} \to (\mathcal{G}, K_{\mathcal{T}}), \quad \text{where} \quad \sum_b K_{\mathcal{T}}(w,b) = \sum_w K_{\mathcal{T}}(w,b) = 0$$





$$\frac{K_{\mathcal{T}}(w,b_k)}{K_{\mathcal{T}}(w,b_1)} = (-1)e^{i\theta_1}\cdot r,$$

with $r \in \mathbb{R}_+$

Then $K_{\mathcal{T}}$ is a Kasteleyn matrix.

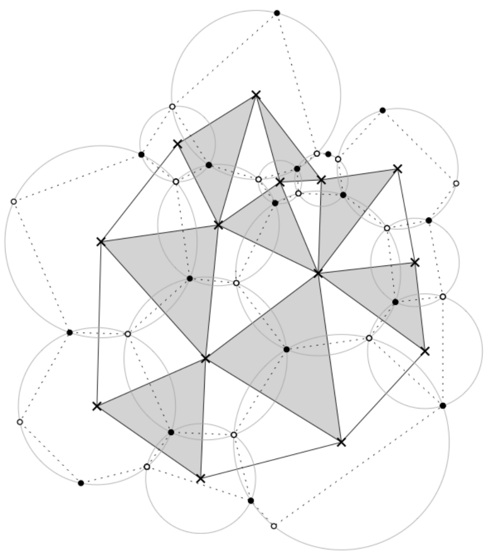
$\boxed{\text{Kasteleyn sign condition}}$ $\longleftrightarrow$ $\boxed{\text{angle condition}}$

$\prod \frac{K_{\mathcal{T}}(w_i,b_i)}{K_{\mathcal{T}}(w_{i+1},b_i)} \in (-1)^{k+1}\mathbb{R}_+$ $\qquad$ $\sum \text{white} = \pi \mod 2\pi$

$$(-1)^k \cdot e^{i\pi} = (-1)^{k+1}$$

(Centers of) circle pattern:

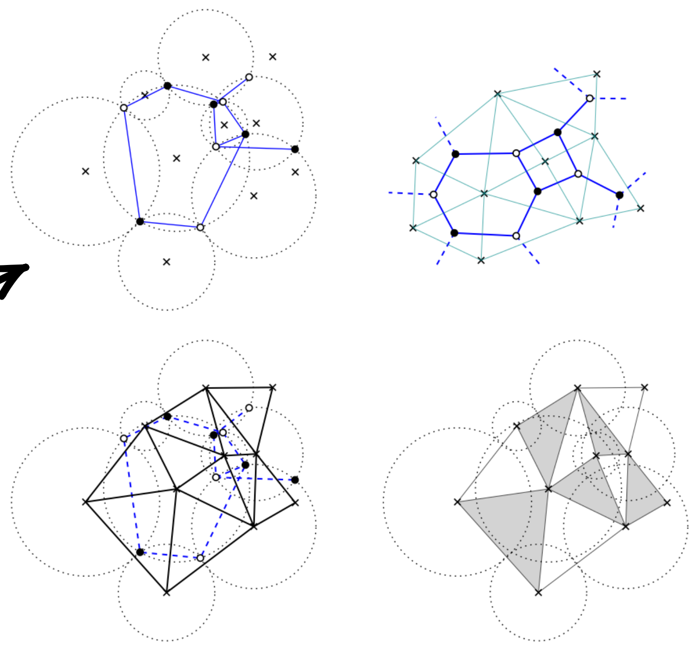
[Kenyon, Lam, Ramassamy, R.]



Circle pattern realisations with an embedded dual, where the dual graph is the graph of circle centres.

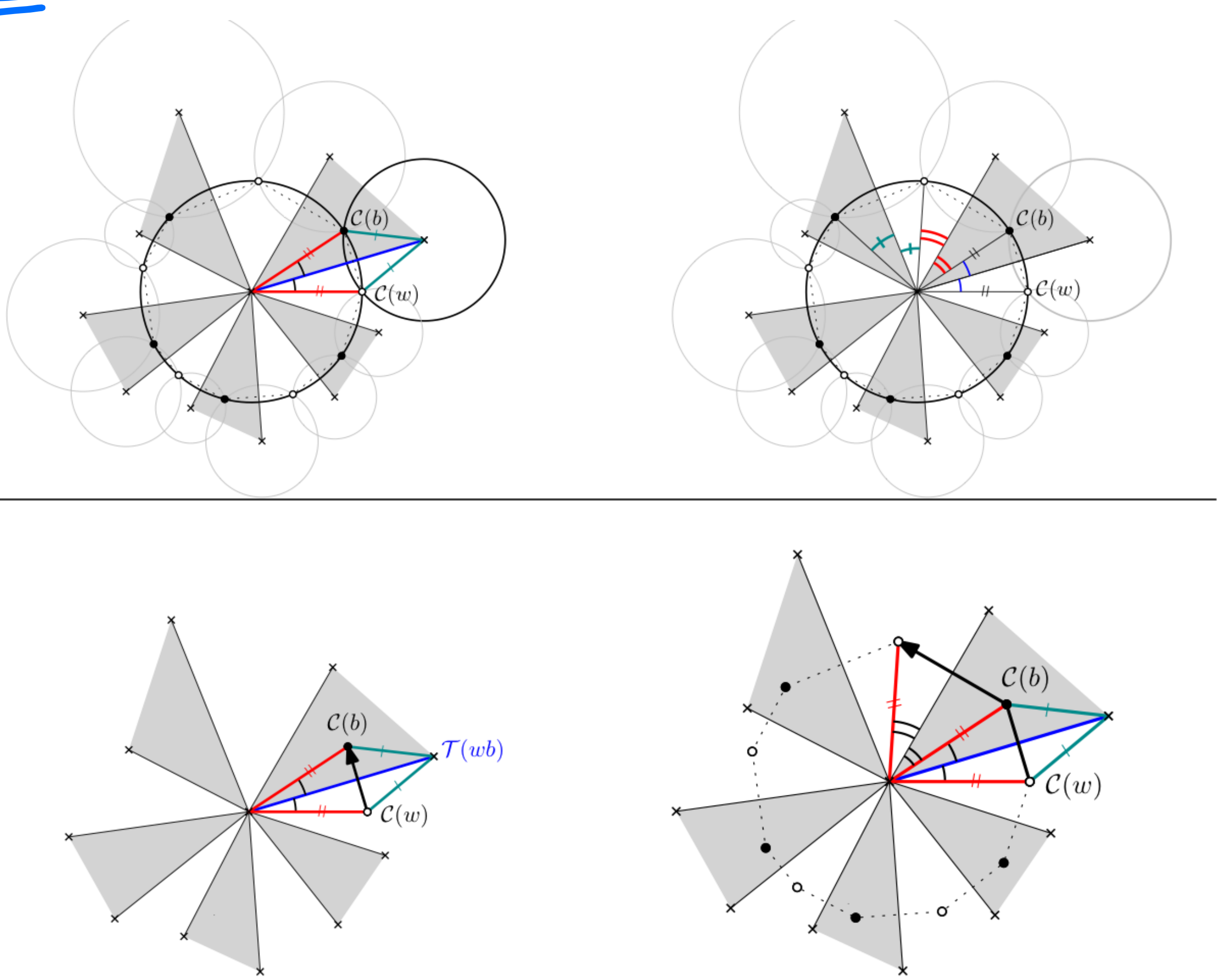
(!) Circle patterns themselves are not necessarily embedded.

A circle pattern realization with an embedded dual

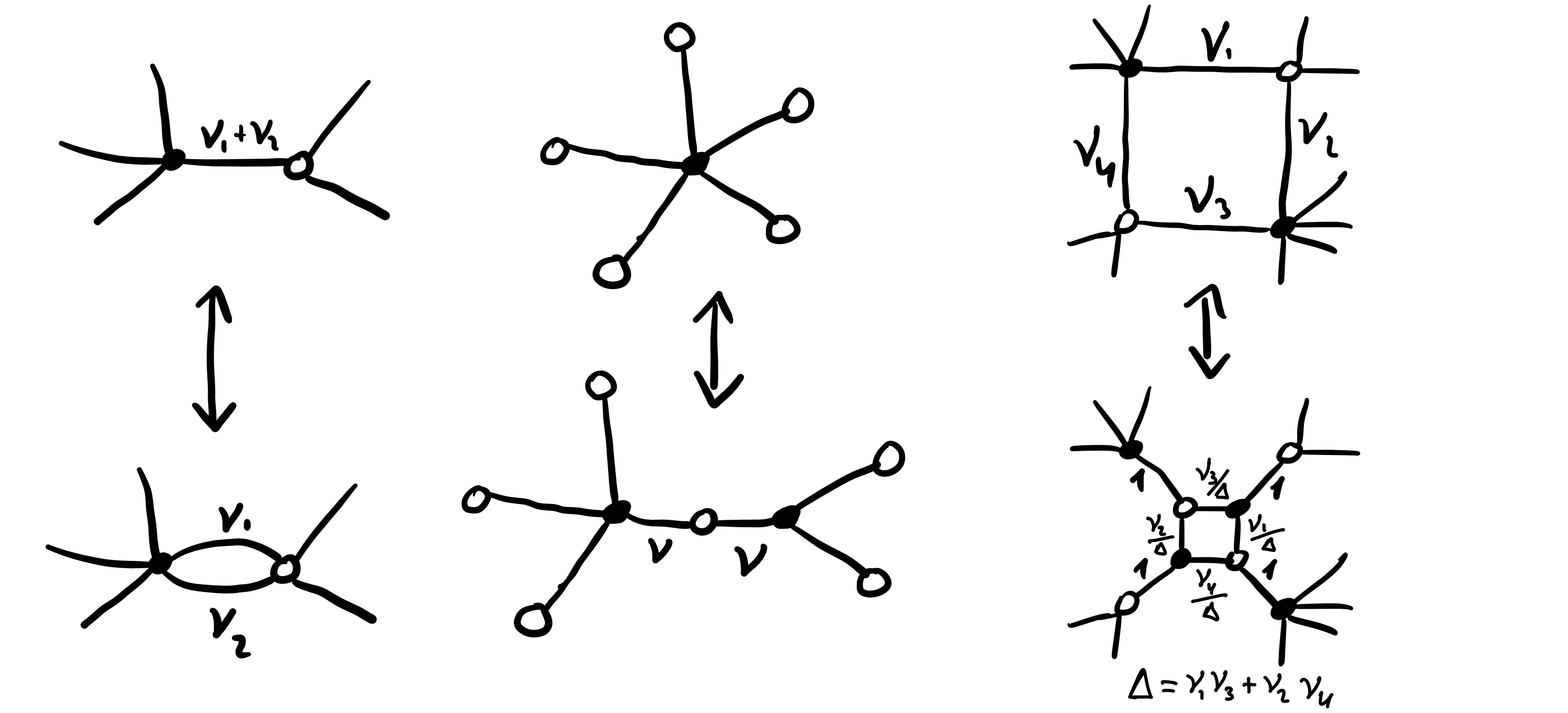


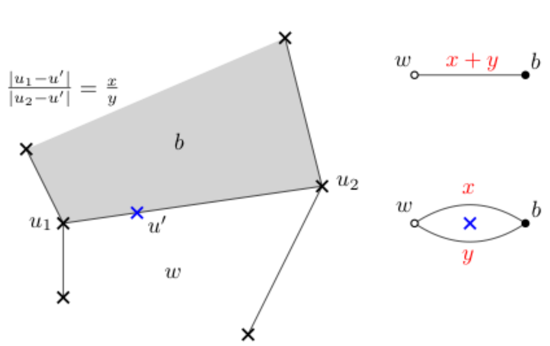
Proposition: Embeddings of the dual graph as centers of a circle pattern are in bijection with t-embeddings.

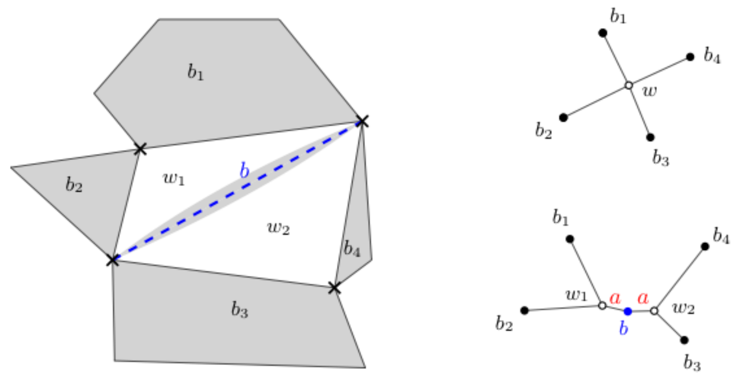
Proof:

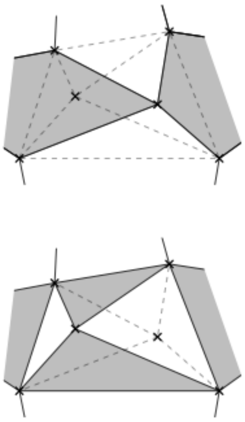


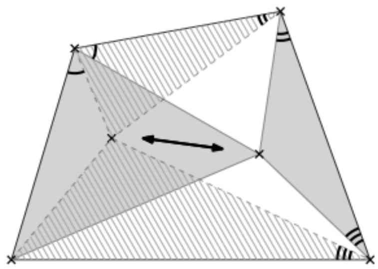
Elementary transformations preserving dimer measure

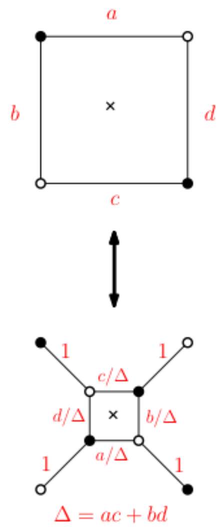










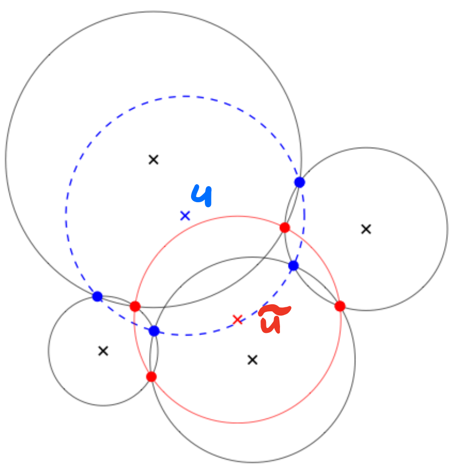


[Kenyon, Lam, Ramassamy, R.]:

T-embeddings of $\mathcal{G}^*$ are preserved under elementary transformations of $\mathcal{G}$.

Spider move in terms of circle patterns:

Miquel's six circles theorem: if five circles share four triple-points of intersection then the remaining four points of intersection lie on a sixth circle.



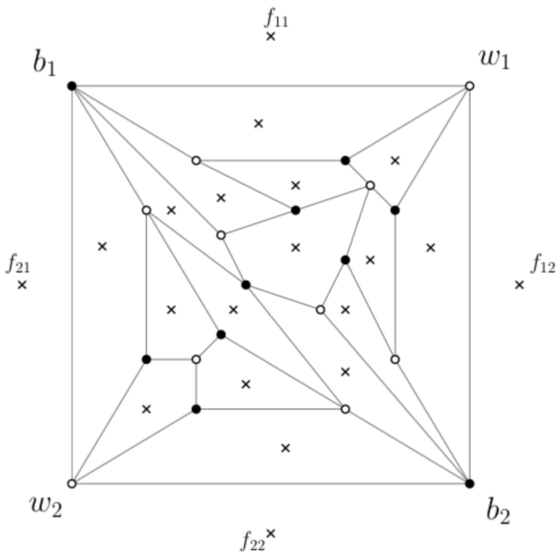
Central move $u \mapsto \tilde{u}$

$$\frac{(u_2-u)(u_4-u)}{(u_1-u)(u_3-u)} = \frac{(u_2-\tilde{u})(u_4-\tilde{u})}{(u_1-\tilde{u})(u_3-\tilde{u})}$$

$$X_u := -\frac{(u_2-u)(u_4-u)}{(u_1-u)(u_3-u)}$$

$$\tilde{u} = \frac{(u_2+u_4) + X_u(u_1+u_3)}{1+X_u}$$

Coulomb gauge for finite planar graphs



Def: Functions $G : W \to \mathbb{C}$ and $F : B \to \mathbb{C}$ are said to give Coulomb gauge for $\mathcal{G}$ if for all internal white vertices w

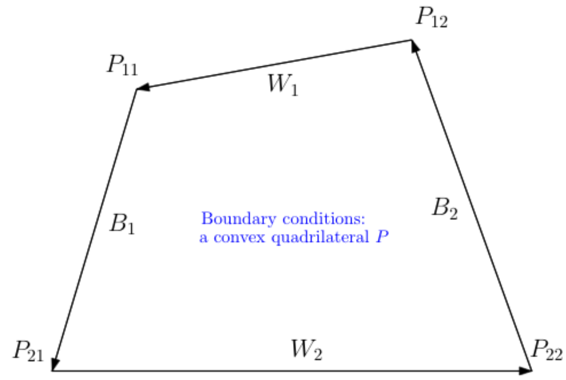
$$\sum_b G(w) K_{wb} F(b) = 0,$$

and for all internal black vertices b

$$\sum_w G(w) K_{wb} F(b) = 0.$$

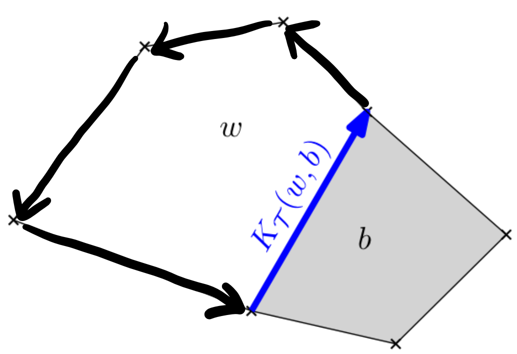
$$\sum_w G(w) K_{wb_i} F(b_i) = B_i$$

$$\sum_b G(w_i) K_{w_i b} F(b) = -W_i.$$



Recall: $\sum_{b\sim w} K_{\mathcal{T}}(w,b) = 0 \Rightarrow \sum_b G(w)\cdot K_{wb} F(b) = 0$

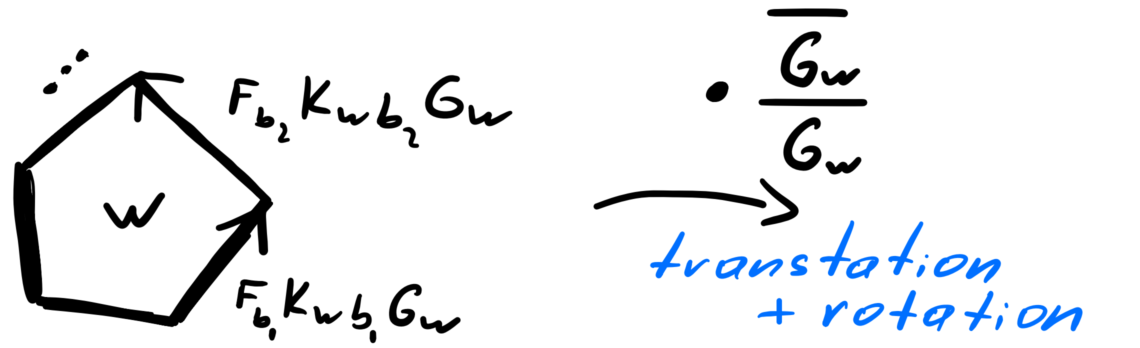
$K_{\mathcal{T}}(w,b) = F(b) K_{\mathbb{R}}(w,b) G(w)$

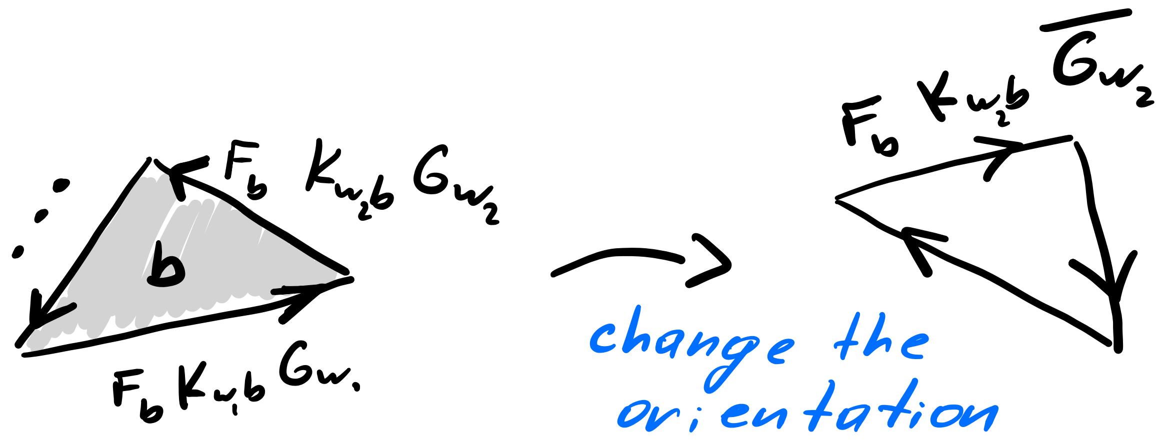


Coulomb gauges <-> t-embedding

$$\boxed{d\mathcal{T}(wb^*) = F(b) K_{wb} G(w)}$$

What if we consider $F_b K_{wb} \overline{G_w}$ instead?





$\boxed{\text{Define } d\theta(wb^*) = F_b K_{wb} \overline{G_w} \qquad \theta(G^*) \text{ is an origami map}}$

// Fold a t-embedding along each edge //