Mini Course: Dimers and Embeddings
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1 Exercise Session One

1. Proof of Kasteleyn's theorem

Let G be a weighted planar bipartite graph, with an edge weight function $\nu : E(G) \to \mathbb{R}_{>0}$.

(a) Show that there exists a choice of *real Kasteleyn signs* for G: There exists signs τ_e for each edge e such that

$$\tau_e = \pm 1 \quad \text{and} \quad \frac{\tau_{e_1}}{\tau_{e_2}} \cdot \ldots \cdot \frac{\tau_{e_{2k-1}}}{\tau_{e_{2k}}} = (-1)^{k+1}$$

around each face of degree $2k$ with boundary edges $e_1, e_2, \ldots, e_{2k}$ in the counterclockwise order.

(b) Let τ_e be Kasteleyn signs on edges. Show that for any simple loop $e_1, e_2, \ldots, e_{2k}$ with l points inside the loop the following holds

$$\frac{\tau_{e_1}}{\tau_{e_2}} \cdot \ldots \cdot \frac{\tau_{e_{2k-1}}}{\tau_{e_{2k}}} = (-1)^{k+l-1}.$$

(c) Assume we have a choice of Kasteleyn signs (not necessary real), and consider the *Kasteleyn matrix*, the matrix whose rows are indexed by black vertices and columns by white vertices, and defined by

$$K(w,b) = \begin{cases} \tau_e \nu(e) & \text{if } (wb) = e \text{ is an edge of G} \\ 0 & \text{otherwise} \end{cases}.$$

Prove that

$$|\det K| = Z,$$

where Z is the *dimer model partition function*. I.e. $Z = \sum_{\text{matchings } M} \left(\prod_{e \in M} \nu(e) \right)$.

2. Local statistics. Let K be a Kasteleyn matrix of a weighted, planar, bipartite graph (G, ν) carrying a dimer model. Show that for any finite set of edges $e_1 = (w_1 b_1), \ldots, e_k = (w_k b_k)$, the probability of seeing these edges in a random perfect matching M is given by the corresponding minor of the inverse Kasteleyn:

$$\mathbb{P}(e_1, \ldots, e_k \in M) = \prod_{i=1}^{k} K(w_i, b_i) \det \left(K^{-1}(b_i, w_j) \right)_{i,j=1}^{k}.$$

Hint: Use that

$$\frac{|\det K_{(W \setminus \{w_j\}_{j=1}^k) \times (B \setminus \{b_j\}_{j=1}^k)}|}{|\det K|} = |\det \left(K^{-1}(b_i, w_j) \right)_{i,j=1}^{k}|.$$

3. Number of tilings of a rectangle. Prove that the number of domino tilings of an $M \times N$ rectangle is given by the product

$$\left(\prod_{p=1}^{M}\prod_{q=1}^{N} 4\left(\cos^2\left(\frac{\pi p}{M+1}\right)+\cos^2\left(\frac{\pi q}{N+1}\right)\right)\right)^{\frac{1}{4}}.$$

Hint: Diagonalize the operator

$$A = \begin{pmatrix} 0 & K \\ K^T & 0 \end{pmatrix}$$

defined in the lectures and apply Kasteleyn's theorem.

4. Proof of Thurston's Theorem Recall that Thurston's theorem states:

Theorem (Thurston)**.** *A simply-connected domain Ω on the square lattice is tileable iff both conditions hold:*

1) *The height function $h|_{\partial\Omega}$ on the boundary vertices is well defined (i.e. the increments around the boundary add up to zero).*
2) *For all vertices $u, v \in \partial\Omega$*

$$h(v) - h(u) \leq d(u,v),$$

where $d(u,v)$ is an edge length of the shortest positive oriented path from u to v within $\overline{\Omega} = \Omega \cup \partial\Omega$ on $\vec{\mathbb{Z}}^2$. Recall that $\vec{\mathbb{Z}}^2$ is a directed graph on the square lattice with checkerboard colored faces such that around each black face the edges oriented clockwise.

In this exercise we will prove Thurston's theorem.

(a) **Proof of $\Longrightarrow$:** Show that for a simply-connected tileable domain on the square lattice the corresponding height function satisfy 1) and 2).

(b) Each positively oriented loop in $\overline{\Omega}$ (i.e. moving along directed edges on $\vec{\mathbb{Z}}^2$) has length divisible by 4.

(c) Assume h is a function defined on boundary vertices and satisfying 1) and 2). Let us define the "maximal height function" $\tilde{h}$ as follows:

$$\tilde{h}(v) = \min_{v' \in \partial\Omega} \left(h(v') + d(v',v)\right).$$

Prove the following lemmas:

Lemma 1.1. *Along each oriented edge $\overrightarrow{uv}$, the following holds*

$$\begin{cases} \tilde{h}(v) \geq \tilde{h}(u) - 3 \\ \tilde{h}(v) \leq \tilde{h}(u) + 1 \end{cases}.$$

Lemma 1.2. *For each oriented edge $\overrightarrow{uv}$, one has*

$$\tilde{h}(v) - \tilde{h}(u) = 1 \mod 4.$$

Hint: To prove the second lemma use part (b).

(d) **Proof of $\Longleftarrow$:** Show that Lemmas 1.1 and 1.2 imply that $\tilde{h}$ satisfies local rules, i.e. corresponds to a tiling.