Mini Course: Dimers and Embeddings

Marianna Russkikh TA: Matthew Nicoletti

Exercise Session Two: Discrete harmonic and holomorphic functions

Below $\Omega \subset \mathbb{C}$ denotes an open simply connected subset.

Reminder:

- A function u is harmonic on Ω iff $\Delta u(z) = u_{xx} + u_{yy} = 0$ for any $z \in \Omega$.
- A form $\omega = P\,dx + Q\,dy$ is called *closed* if for any loop γ we have $\int_\gamma \omega = 0$. In this case, we can define a primitive F of ω (i.e., a function F such that $dF = \omega$) by letting $F(z) = \int_{z_0}^{z} \omega$, where $z_0 \in \Omega$ is some fixed point and $\int_{z_0}^{z}$ denotes the integral along any path in Ω connecting z_0 and z.

1. Harmonic conjugate

(a) Let $u : \Omega \to \mathbb{R}$ be a harmonic function. Show that $d^*u := u_x\,dy - u_y\,dx$ is a closed form.
[Use Green's theorem: for any $P, Q \in C^1(\Omega)$ one has $\int_{\partial\Omega} P\,dx + Q\,dy = \iint_\Omega \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$.]

(b) In the setup of (1a), define v to be the primitive of d^*u. Observe that ∇v is equal to ∇u rotated by $\pi/2$ counterclockwise everywhere.

(c) Show that $f := u + iv$ is a holomorphic function.

(d) Check that for any function $f : \Omega \to \mathbb{C}$ one has $4\partial\bar{\partial} f = 4\bar{\partial}\partial f = \Delta f$.

2. Discrete harmonic conjugate

A function $u : \mathbb{Z}^2 \to \mathbb{R}$ is called discrete harmonic at $b \in \mathbb{Z}^2$ if

$$\Delta_{\text{discr}} u(b) = \frac{u(b+1) + u(b+i) + u(b-1) + u(b-i) - 4u(b)}{4} = 0.$$

(a) Check that if $u \in C^2(\mathbb{C})$, then for any $b \in \mathbb{C}$

$$\frac{u(b+\varepsilon) + u(b+i\varepsilon) + u(b-\varepsilon) + u(b-i\varepsilon) - 4u(b)}{4} = \frac{\varepsilon^2}{4}\Delta u + o(\varepsilon^2),$$

i.e., Δ_{discr} approximates Δ in a certain sense.

(b) Given an oriented edge $(b_1 b_2)$ of $\mathbb{Z}^2$, denote by $(b_1^* b_2^*)$ the oriented edge of $\left(\mathbb{Z} + \frac{1}{2}\right) \times \left(\mathbb{Z} + \frac{1}{2}\right)$ which has the first vertex (here b_1) to its right. Define a 1-form on oriented edges of $\left(\mathbb{Z} + \frac{1}{2}\right) \times \left(\mathbb{Z} + \frac{1}{2}\right)$ by

$$\omega(b_1^* b_2^*) := u(b_2) - u(b_1).$$

Show that ω is a closed form (sums to zero around any loop in the dual graph) if u is discrete harmonic.

(c) Define a function $v : \left(\mathbb{Z}+\frac{1}{2}\right)\times\left(\mathbb{Z}+\frac{1}{2}\right)\to\mathbb{R}$ to be the primitive of ω, which means that the equality

$$v(b_1^*) - v(b_2^*) = \omega(b_1^* b_2^*)$$

holds for any adjacent vertices b_1^* and b_2^* of $\left(\mathbb{Z}+\frac{1}{2}\right)\times\left(\mathbb{Z}+\frac{1}{2}\right)$. Show that v is discrete harmonic.

(d) Let u and v be defined as above. Let $B := \mathbb{Z}^2 \cup \left(\mathbb{Z}+\frac{1}{2}\right)\times\left(\mathbb{Z}+\frac{1}{2}\right)$ and define a function $f : B \to \mathbb{R}\cup i\mathbb{R}$ by

$$f(b) = \begin{cases} u(b) & \text{if } b \in \mathbb{Z}^2 \\ iv(b) & \text{if } b \in \left(\mathbb{Z}+\frac{1}{2}\right)\times\left(\mathbb{Z}+\frac{1}{2}\right). \end{cases}$$

Let us define discrete operators ∂_{discr} and $\bar{\partial}_{\text{discr}}$ by the formulas:

$$[\partial_{\text{discr}} f](w) = \frac{1}{2}\left(\frac{f(w+\frac{1}{2}) - f(w-\frac{1}{2})}{2} + \frac{f(w+\frac{i}{2}) - f(w-\frac{i}{2})}{2i}\right),$$

$$[\bar{\partial}_{\text{discr}} f](w) = \frac{1}{2}\left(\frac{f(w+\frac{1}{2}) - f(w-\frac{1}{2})}{2i} + \frac{f(w+\frac{i}{2}) - f(w-\frac{i}{2})}{2}\right),$$

Show that $[\bar{\partial}_{\text{discr}} f](w) = 0$ for all $w \in W := (\mathbb{Z}\times(\mathbb{Z}+\frac{1}{2}))\cup((\mathbb{Z}+\frac{1}{2})\times\mathbb{Z})$.

(e) Suppose that $f \in C^1(\mathbb{C})$. Show that

$$\frac{1}{2}\left(\frac{f(w+\frac{\varepsilon}{2}) - f(w-\frac{\varepsilon}{2})}{2i} + \frac{f(w+i\frac{\varepsilon}{2}) - f(w-i\frac{\varepsilon}{2})}{2}\right) = -i\frac{\varepsilon}{2}\bar{\partial} f + o(\varepsilon),$$

i.e., $\bar{\partial}_{\text{discr}}$ approximates $\bar{\partial}$.

Definition: we call a pair $f := (u, iv)$ a holomorphic function and associate u with the real part of f and v with its imaginary part.

(f) Show that $4[\partial_{\text{discr}}\bar{\partial}_{\text{discr}} f](b) = 4[\bar{\partial}_{\text{discr}}\partial_{\text{discr}} f](b) = \Delta_{\text{discr}} f(b)$.