Mini Course: Dimers and Embeddings

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Exercise Session Four: t-embeddings and t-holomorphicity

Definition: *gauge equivalence*
Two weight functions ν_1, ν_2 are said to be gauge equivalent if there is a function $F : V(G) \to \mathbb{R}$ such that for any edge vv', $\nu_1(vv') = F(v)F(v')\nu_2(vv')$. Gauge equivalent weights define the same probability measure μ.

Definition: *origami square root function*
A function $\eta : V(G) = B \cup W \to \mathbb{T}$ is said to be an *origami square root function* if it satisfies the identity

$$\overline{\eta}_b \overline{\eta}_w = \frac{dT(bw^*)}{|dT(bw^*)|}$$

for all pairs (b,w) of white and black neighbouring faces of G^*.

Definition: *t-white-holomorphicity*
Let η be an origami square root function. A function $F : B \to \mathbb{C}$ is called t-white-holomorphic at $w \in W$ if

$$\begin{cases} F(b) \in \eta_b \mathbb{R} & \forall b \in B, b \sim w, \\ \oint_{\partial w} F\,dT = 0 & (\iff (K_T F)(w) = \sum_{b \sim w} K_T(w,b)F(b) = 0). \end{cases}$$

1. **Gauge equivalence.** For a planar bipartite graph, show that two weight functions are gauge equivalent if and only if their face weights are all equal, where the face weight X_f of a face f with vertices (in the counterclockwise order) w_1, b_1, ..., w_k, b_k is the "alternating product" of its edge weights:

$$X_f := \prod_{j=1}^{k} \frac{\nu(w_j b_j)}{\nu(w_{j+1} b_j)}$$

(here we index cyclically so we think of $k+1 = 1$).

Hint: To show "same face weights $\implies$ gauge equivalence", define an appropriate discrete one form on the (directed) edges of the graph, and argue that it can be "integrated" to a function.

2. **Origami map.** Let η be an origami square root function.

   a) Show that on edges

$$\eta_w^2 dT(bw^*) = \eta_w \overline{\eta}_b |dT(bw^*)| = \overline{\eta}_b^2 \overline{dT(bw^*)}$$

   b) Define a smooth differential form dO on the union of faces of the t-embedding T by

$$dO(z) := \begin{cases} \eta_w^2 dz & \text{if } z \text{ in the white face } T(w), \\ \overline{\eta}_b^2 d\bar{z} & \text{if } z \text{ in the black face } T(b). \end{cases}$$

   Show that this definition is consistent along edges of T, and that dO is a closed form (by which we mean it integrates to 0 around closed loops) inside the domain covered by T.

3. **t-holomorphicity.** Assume that all white faces of the t-embedding T are triangles, and let $\eta : W \cup B \to \mathbb{T}$ be an origami square root function. Let $F : B \to \mathbb{C}$ be a function such that $\forall b \in B$, $F(b) \in \eta_b \mathbb{R}$. Show that F can be extended to a white vertex w such that $Proj(F(w); \eta_b \mathbb{R}) = F(b)$ for all $b \sim w$, if and only if

$$\oint_{\partial w} F dT = 0.$$

4. **Closed forms.** Below, if we multiply a function on (primal) vertices by a discrete one form on G^*, we will use the upper subscript to indicate which primal vertex adjacent to the edge we use to evaluate the function. So if F is a function on all primal vertices, the form $(F^\bullet dT)(bw^*) := F^\bullet(b) dT(bw^*)$, and $(F^\circ dT)(bw^*) := F^\circ(w) dT(bw^*)$.

   a) Let U be a simply connected region in the domain of a tembedding and F be a t-white-holomorphic function on a punctured region U. Then, on edges not adjacent to boundary white faces,

   $$2F^\bullet dT = F^\circ dT + \overline{F^\circ}\,\overline{dO}$$

   and $F_w^\bullet dT$ is a closed form in U away from the boundary (i.e., the integral over any closed contour γ running over interior edges vanishes).

   b) If F_b and F_w are respectively t-black- and t-white-holomorphic functions on some region U, then, on edges not adjacent to boundary faces, the identity

   $$F_w^\bullet F_b^\circ dT = \frac{1}{2} \mathrm{Re}\left(F_w^\circ F_b^\bullet dT + F_w^\circ \overline{F_b^\bullet} dO\right)$$

   holds and the form $F_w^\bullet F_b^\circ dT$ is closed in U away from the boundary.

5. **Origami map of perfect t-embeddings.** Check that all boundary segments of a perfect t-embedding are mapped onto the same line under an origami map.