Virginia Integrable Probability Summer School 2024

Tagged particle fluctuations in TASEP with a moving wall

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joint work with Patrik Ferrari
arXiv:2403.05366

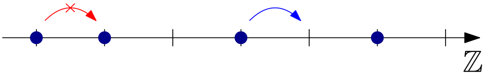
Dynamics of TASEP I

Totally Asymmetric Simple Exclusion Process (TASEP):

- Configurations:

$$\eta = (\eta(i))_{i\in\mathbb{Z}},\ \eta(i) = \begin{cases} 1 \text{ if } i \text{ is occupied,} \\ 0 \text{ if } i \text{ is empty.} \end{cases}$$

- Dynamics: Independently of each other, particles try to jump to the right by one step with rate 1. A jump occurs iff the arrival site is empty.

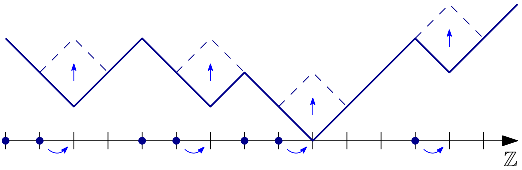


Dynamics of TASEP II

Interpretations:

- Particle positions: $x_n(t)$ is the position of the n-th particle at time t, with $n \in \mathbb{Z}$. We set $x_{n+1}(t) < x_n(t)$.
- Height function: TASEP is a random growth model in the KPZ universality class, with height function given by

$$h(0,0) = 0, \;\; h(x+1,t) - h(x,t) = 1 - 2\eta_t(x).$$



One-point distributions in the large time limit I

- Spatial / height fluctuations in $\mathcal{O}(T^{1/3})$, spatial correlations in $\mathcal{O}(T^{2/3})$
- [Matetski, Quastel, Remenik '21] showed convergence to the KPZ fixed point
- Under mild assumptions on $h(\cdot, 0)$, including

$$\lim_{\ell \to \infty} -\ell^{-1/2} h(2\ell x, 0) = h_0(x),$$

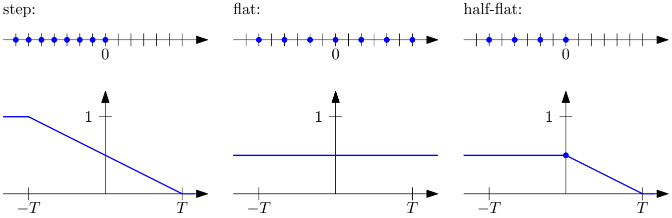
it holds

$$\lim_{T \to \infty} \mathbb{P}\left(\frac{h(0, T) - \frac{T}{2}}{-(\frac{T}{2})^{1/3}} \leq s \right) = \mathbb{P}\left(\sup_{\tau \in \mathbb{R}} \{ \mathcal{A}_2(\tau) - \tau^2 + h_0(\tau) \} \leq s \right)$$

for $\mathcal{A}_2$ being an Airy$_2$ process.

One-point distributions in the large time limit II

Initial conditions and macroscopic density profiles:



Convergence of finite-dimensional distributions:

Initial condition	Limit process	One-point distribution
step	Airy$_2$ process $\mathcal{A}_2$	GUE Tracy-Widom F_{GUE}
flat	Airy$_1$ process $\mathcal{A}_1$	GOE Tracy-Widom F_{GOE}
half-flat	Airy$_{2\to1}$ process $\mathcal{A}_{2\to1}$	$F_{2\to1;x}$

[Prähofer, Spohn '02], [Borodin, Ferrari, Prähofer, Sasamoto '07], [Borodin, Ferrari, Sasamoto '08]

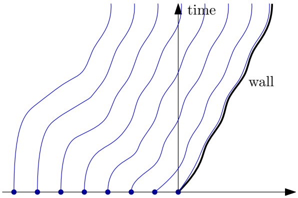
Our model

TASEP with step initial condition and wall constraint:

- Position of wall at time $t \geq 0$: $f(t)$; f non-decreasing with $f(0) = 0$.
- Particle positions: $x_n^f(t)$.
- Step initial condition: $x_n^f(0) = -n + 1,\ n \in \mathbb{N}$.
- First particle blocked by the wall: $x_1^f(t) \leq f(t),\ t \geq 0$.

Question: What is the limit distribution of (rescaled) $x^f_{\alpha T}(T)$, given influences of the wall around two macroscopic times $\alpha_0 T, \alpha_1 T$, for $\alpha \in (0,1)$, $\alpha_0 \neq \alpha_1 \in (\alpha, 1)$?

The case of one wall influence has been discussed by [Borodin, Bufetov, Ferrari '24].



Particle trajectories

TASEP with step initial condition

Let $x(t)$ denote a TASEP with step initial condition.
Macroscopic position:

$$x_{\alpha T}(\beta T) \simeq \begin{cases} \left(1 - 2\sqrt{\frac{\alpha}{\beta}}\right)\beta T, & \beta \in (\alpha, 1], \\ -\alpha T, & \beta \in [0, \alpha]. \end{cases}$$

Lemma 1 (e.g. [Borodin, Péché '08],[Ferrari, Occelli '18])

Let $\mathcal{A}_2$ be an Airy$_2$ process. For $t = \beta T - \tilde{c}_\beta \tau T^{2/3}$ with $\beta \in (\alpha, 1)$ and some constants $\tilde{c}_\beta, c_\beta > 0$, it holds

$$\lim_{T \to \infty} \frac{x_{\alpha T}(t) - \left(1 - 2\sqrt{\frac{\alpha T}{t}}\right) t}{-c_\beta T^{1/3}} = \mathcal{A}_2(\tau)$$

weakly with respect to the uniform topology on bounded sets.

A finite-time identity I

Scaling: For a non-trivial wall influence, we expect

$$x^f_{\alpha T}(T) \simeq \xi T \text{ for some } \xi \in (-\alpha, 1 - 2\sqrt{\alpha}).$$

Lemma 2 (Borodin, Bufetov, Ferrari '24)

For any $S \in \mathbb{R}$,

$$\mathbb{P}(x^f_{\alpha T}(T) \geq \xi T - ST^{1/3}) = \mathbb{P}(x_{\alpha T}(t) \geq \xi T - f(T-t) - ST^{1/3} \ \forall t \in [0, T]).$$

A finite-time identity II

$$\mathbb{P}(x^f_{\alpha T}(T) \geq \xi T - ST^{1/3}) = \mathbb{P}(x_{\alpha T}(t) \geq \xi T - f(T-t) - ST^{1/3}\ \forall t \in [0,T])$$

Assumptions:

(A1) $\xi T - f(T-t) \simeq \left(1 - 2\sqrt{\frac{\alpha T}{t}}\right) t - c_\beta g_\beta(\tau) T^{1/3}$ for $t = \beta T - \tilde{c}_\beta \tau T^{2/3}$, $|\tau| \leq \varepsilon T^{1/3}$, $\beta \in \{\alpha_0, \alpha_1\}$, $\varepsilon > 0$ fixed. Let g_β be piecewise continuous with $g_\beta(\tau) \geq -M + \frac{\tau^2}{2}$ for some constant M.

(A2) $\xi T - f(T-t) \ll \mathrm{LLN}(x_{\alpha T}(t))$ else.

Limit of one-point distributions

Theorem 1 (Ferrari, G.'24)

Under Assumptions A1 and A2, it holds

$$\lim_{T\to\infty} \mathbb{P}(x^f_{\alpha T}(T) \geq \xi T - ST^{1/3}) = \prod_{\beta\in\{\alpha_0,\alpha_1\}} \mathbb{P}\Big(\sup_{\tau\in\mathbb{R}}\{\mathcal{A}_2(\tau) - g_\beta(\tau)\} \leq c_\beta^{-1} S\Big).$$

Variational formulas:

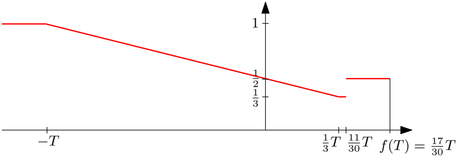
- [Johansson '03]: $F_{\mathrm{GOE}}(2^{2/3}s) = \mathbb{P}\Big(\sup_{\tau\in\mathbb{R}}\{\mathcal{A}_2(\tau) - \tau^2\} \leq s\Big)$.
- [Quastel, Remenik '13]: $F_{2\to 1;0}(s) = \mathbb{P}\Big(\sup_{\tau\leq 0}\{\mathcal{A}_2(\tau) - \tau^2\} \leq s\Big)$.

Example: A piecewise linear function f

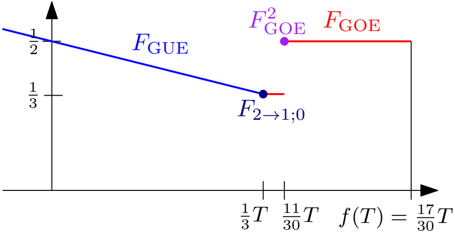
$$f(t) = \begin{cases} \frac{2}{3}t, & t \in [0, 0.35T), \\ \frac{1}{15}T + \frac{1}{2}t, & t \in [0.35T, T]. \end{cases}$$

$g_{\alpha_0}(\tau) = g_{\alpha_1}(\tau) = \tau^2$, $\alpha_0 = 0.4$, $\alpha_1 = 0.9$ and $x^f_{0.1T}(T) \simeq \frac{11}{30}T$.

Macroscopic density profile:



Limit distributions:



Comparison to the case of one wall influence

Similarities:

- Use of the identity

$$\mathbb{P}(x^f_{\alpha T}(T) \geq \xi T - ST^{1/3}) = \mathbb{P}(x_{\alpha T}(t) \geq \xi T - f(T-t) - ST^{1/3} \,\forall t \in [0, T]).$$

- Assumptions on f.
- Convergence of the tagged particle process in TASEP with step initial condition.

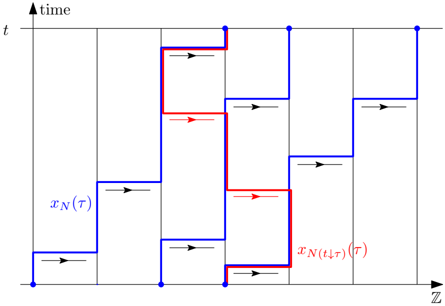
Difference: For multiple wall influences, we require an asymptotic decoupling of particle positions.

Theorem 2

Let $I_\beta = [\beta T - \kappa T^{2/3}, \beta T + \kappa T^{2/3}]$ for $\kappa > 0$ fixed. Then, $(x_{\alpha T}(t))_{t \in I_{\alpha_0}}$ and $(x_{\alpha T}(t))_{t \in I_{\alpha_1}}$ are asymptotically independent.

Asymptotic decoupling I

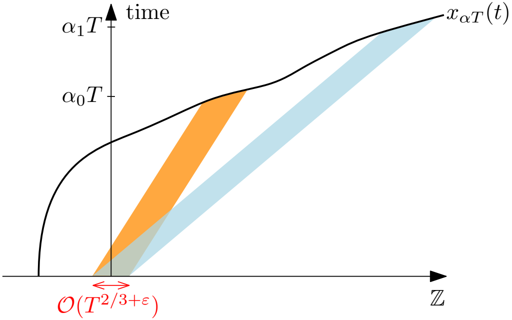
Step 1: Localization of space-time correlations



- $x_{N(t\downarrow\tau)}(\tau)$ denotes the backwards path starting at $x_N(t)$.
- It depicts the space-time correlations of the tagged particle.
- We localize it in a deterministic region.

Asymptotic decoupling II

Step 1: Localization of space-time correlations



$(x_{\alpha T}(t))_{t \in I_{\alpha_0}}$ and $(x_{\alpha T}(t))_{t \in I_{\alpha_1}}$ asymptotically only depend on the orange resp. blue region. Their widths are in $\mathcal{O}(T^{2/3+\varepsilon})$ for any fixed $\varepsilon > 0$.

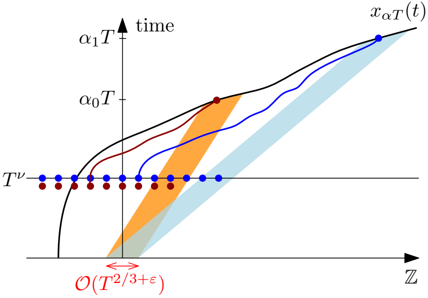
Asymptotic decoupling III

Step 2: Slow decorrelation

For $\nu \in (0,1)$, $x_{\alpha T}(\beta T)$ asymptotically does not depend on events during $[0, T^\nu]$:

$$x_{\alpha T}(\beta T) = x_{\alpha T - \frac{\alpha}{\beta} T^\nu}^{\mathrm{step}, x_{\frac{\alpha}{\beta} T^\nu}(T^\nu)}(T^\nu, \beta T) + o(T^{1/3})$$

- $x^{\mathrm{step},x_n(T^\nu)}(T^\nu, t)$: TASEP with step initial condition shifted by $x_n(T^\nu)$ and starting at time T^ν; coupled to $x(t)$.
- Same regions of correlations; for $\nu > \frac{2}{3} + \varepsilon$, they are disjoint above T^ν. This implies the decoupling.



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