INTRODUCTION TO THE KPZ FIXED POINT: NOTES FOR VIRGINIA INTEGRABLE PROBABILITY SUMMER SCHOOL 2024

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Contents

1. Introduction

This course studies various mathematical models of surface growth. If you put the edge of a sheet of paper in ink, you would observe that the interface between the dry and wet regions grows gradually, with a random, jagged looking interface. The interface grows in a fairly simple manner macroscopically (e.g. in the form of a line or circle) but the fluctuation grows in time. This type of phenomenon has been termed *kinetic roughening* and here our main interest is to understand how the fluctuation grows and what kinds of properties these fluctuations possess. There are many examples of this kind of growth, such as growth of bacterial colonies, forest fires, etc. This is clearly a dynamic process and is an example of a nonequilibrium phenomenon. Compared to systems in thermal equilibrium, our understanding of nonequilibrium systems is still unsatisfactory which is one of the reasons the progress on KPZ is important.

The macroscopic part of the interface growth might be best modelled by a Hamilton-Jacobi PDE

$$\partial_t h = H(\partial_x h), \qquad h(0,x) = h_0(x) \tag{1}$$

We want to add noise to this to make a prototype model. It is not so clear how exactly we should do this. At this point (and as we will see, at many other points in the discussion) it is a good idea to look at simple discrete examples to guide us. A famous motivating example is *ballistic deposition*. This is a continuous time surface growth model on an integer lattice with the following stochastic rule. Let us denote by $h(x,t) \in \mathbb{Z}$ the height at position $x \in \mathbb{Z}$ and at time t. Attached to each x there is an independent Poisson process $N_x(t)$. At each jump time of the Poisson process, a particle arrives at x and either 1) stacks on top of the particles already there, so $h(x) \mapsto h(x)+1$, or 2) sticks to one of the neighbouring stacks, so $h(x) \mapsto \max\{h(x-1), h(x+1)\}$, whichever is larger. So the rule is that

$$h(x) \mapsto \max\{h(x-1), h(x)+1, h(x+1)\} \tag{2}$$

independently at each site x at rate 1.

This defines the stochastic dynamics of the surface. We will see later how to model such processes as Markov processes. It is fairly easy to see how the surface grows under these rules by using computer simulations. On average, the surface grows linearly in time with a fixed velocity. One also observes that fluctuations around the average grow in time. We are interested in these fluctuations. If there were no influence from the neighboring sites in the above deposition rules, the height at each site would grow independently of other parts. Then, by the central limit theorem, the fluctuation of the surface would grow like $\mathcal{O}(t^{1/2})$ and the scaled height would obey the Gaussian distribution. The question is whether this behavior persists when we have the influence from the neighbors. There are several ways neighbors can influence, and in real systems, particles can actually move about along the interface, producing a smoothing effect which keeps the interface continuous. One finds the $\mathcal{O}(t^{1/2})$ from completely independent piles is a bit of a hoax and with some smoothing it actually grows like $\mathcal{O}(t^{1/4})$.

But in simulations, one observes that the fluctuations grow like $\mathcal{O}(t^{1/3})$ rather than $\mathcal{O}(t^{1/4})$. One would guess that the surfaces at various points have strong correlations and as a result the fluctuation becomes no longer Gaussian. There is something nontrivial going on, which we want to understand. The same exponent is seen in many simulation models, and in real experiments. From the experience in equilibrium statistical physics, we expect that there is some universality.

The influence of the sticky neighbours has two effects actually, which should be distinguished. Firstly, there is a non-linear effect, which is what the $H(\partial_x h)$ is modelling. Secondly, there is an effect of keeping the interface somewhat regular. After all, this dynamics definitely tends to reduce large gradients $|h(x+1)-h(x)|$. The simplest continuum analogue of surface regularization is the heat equation

$$\partial_t h = \partial_x^2 h. \tag{3}$$

So we might think of adding such a term to (1) to get a *viscous* Hamilton-Jacobi equation

$$\partial_t h = H(\partial_x h) + \partial_x^2 h. \tag{4}$$

On top of this we should add the noise. The Poisson processes are independent, so it is reasonable to think that the forcing noise should be independent in space. But also the noise arriving over two non-intersecting ($[s_1, s_2) \cap [t_1, t_2) = \emptyset$) time intervals $N_x(t_2) - N_x(t_1)$ and $N_x(s_2) - N_x(s_1)$ are independent for Poisson processes. Somehow Gaussian processes are the most natural for modelling, so we force the thing with space-time Gaussian white noise, i.e. the Gaussian field $\xi(t, x)$ with (informally)

$$\langle \xi(t, x), \xi(s, y) \rangle = \delta(t - s)\delta(x - y). \tag{5}$$

There is a not-so-small technicality that $\xi(t, x)$ turns out to be a distribution instead of a function. So $\int \phi(t, x)\xi(t, x)dtdx$ makes sense for a test function $\phi(t, x)$, it is a Gaussian random variable with mean zero and the correlations with different ϕ's is

$$E\left[\int \phi_1(t, x)\xi(t, x)dtdx \int \phi_2(t, x)\xi(t, x)dtdx\right] = \int \phi_1(t, x)\phi_2(t, x)dtdx. \tag{6}$$

Exercise 1.1. Let $\mathcal{E}$ be a nice subset of $\mathbb{R}^d$ and e_n, $n = 1, 2, \ldots$ be an orthonormal basis of $L^2(\mathcal{E})$ and $\xi_1, \xi_2, \ldots$ be independent standard Gaussians (i.e. mean zero and variance 1). Show that $\xi = \sum_n \xi_n \mathrm{e}_n$ is Gaussian white noise on $\mathcal{E}$. Why is it called *white*? Show that the space-time white noise rescales as,

$$\xi(t, x) \overset{\text{dist}}{=} \epsilon^{\frac{z+1}{2}} \xi(\epsilon^z t, \epsilon^1 x). \tag{7}$$

Exercise 1.2. Let $\mathcal{E}$ = nice subsets of $\mathbb{R}^d$. A Poisson point process on $\mathcal{E}$ with rate 1 is a (random) map N: subsets of $\mathcal{E} \to \{0, 1, 2, \ldots\}$ such that for each $B \subset \mathcal{E}$, $P(N(B) = n) = \frac{|B|^n}{n!} e^{-|B|}$ and $N(B_1)$ and $N(B_2)$ are independent if B_1 and B_2 are disjoint . For simple functions $\phi(x) = \sum_{i=1}^n a_i \mathbf{1}_{A_i}$ we can define $\int \phi dN = \sum_{i=1}^n a_i N(A_i)$ and this extends to $\phi \in L^2(\mathcal{E})$ by density. Compute the covariance of $\int \phi_1 dN$ and $\int \phi_2 dN$. Find a rescaling under which the Poisson point process on $[0, \infty) \times \mathbb{R}$ converges to the space-time Gaussian white noise.

Exercise 1.3. Let $\mathcal{E} = [0, \infty)$ and ξ be white noise on $\mathcal{E}$. Define $B(t) = \int \mathbf{1}_{[0,t)}(x)\xi(x)dx$. Prove that it has independent increments, i.e. $B(t_{i+1}) - B(t_i)$ are independent if the intervals don't overlap, and that they are Gaussian, mean zero and variance the length of the interval, i.e. $B(t)$ is *Brownian motion.* Now do something similar on $\mathcal{E} = \mathbb{R}$ to get a *two-sided Brownian motion.* Compute the covariance $E[B(s)B(t)]$.

In the next section we will show that Brownian motion paths are continuous.

Exercise 1.4. Let $\mathcal{E}$ be a nice simply connected open set in $\mathbb{R}^d$ with a smooth boundary, let Δ be the Laplace operator on $\mathcal{E}$ with maybe Dirichlet or Neumann boundary conditions, and let e_n, $n = 1, 2, \ldots$ be an orthonormal basis of $L^2(\mathcal{E})$ consisting of eigenfunctions of Δ with corresponding eigenvalues λ_n. Show that $\xi = \sum_n (1 - \lambda_n)^{-1/2} \xi_n \mathrm{e}_n$ is a representation of the *Gaussian free field*, i.e. the Gaussian field whose formal covariance is the Green's function $(1 - \Delta)^{-1}$.

Exercise 1.5. Define the diffusive rescaling operators $\mathcal{S}_\epsilon$ by

$$(\mathcal{S}_\epsilon f)(x) = \epsilon^{1/2} f(\epsilon^{-1} x) \tag{8}$$

Show that the probability measure P on the space of continuous functions $\mathcal{C}_0[0,\infty)$ with $f(0) = 0$ corresponding to Brownian motion is invariant under $\mathcal{S}_\epsilon$. Let $X_1, X_2, \ldots$ be independent identically distributed random variables with mean 0 and variance $\sigma^2 < \infty$. We can make a probability measure P_1 on $\mathcal{C}$ by setting $S(0) = 0$, $S(t) = \sum_{i=1}^n X_i$ when $t = n \in \{1, 2, \ldots\}$, and linear in between. $\mathcal{S}_\epsilon$ takes this to a measure P_ϵ. Then, as $\epsilon \to 0$,

$$P_\epsilon \to P, \tag{9}$$

in the sense of weak convergence of probability measures. Donsker called this the *invariance principle*. Prove that it holds in the sense of finite dimensional distributions, i.e.

$$P_\epsilon(A) \to P(A), \tag{10}$$

whenever B is a set of the form $B = \{f : f(x_1) \in B_1, \ldots, f(x_n) \in B_n\}$ for Borel subsets $B_1, \ldots, B_n$ of $\mathbb{R}$.

Exercise 1.6 (Variation of constants)**.** We'll denote by $x(t) = e^{At}x$ the solution of the equation

$$\dot{x} = Ax, \qquad x(0) = x. \tag{11}$$

Show that

$$y(t) = e^{At}x + \int_0^t e^{A(t-s)} f(s) ds \tag{12}$$

is the solution of the *inhomogeneous equation*

$$\dot{y} = Ay + f, \qquad y(0) = x. \tag{13}$$

Exercise 1.7 (Ornstein-Uhlenbeck)**.** Consider the *Langevin equation*

$$\dot{v} = -\lambda v + \sqrt{D}\xi \tag{14}$$

where v is the $d-$dimensional *velocity process* and ξ is a $\mathbb{R}^d$ valued white noise on $[0,\infty)$. The idea is that the integral of v is a somewhat more realistic model of the physical Brownian motion since it has a well defined velocity. Use variation of constants to solve for v. Show that the resulting process is Gaussian and compute its covariance. Show that as λ becomes small it converges to the mathematicians Brownian motion. Find an appropriate Gaussian for the initial state so that the process is stationary in time. By pushing the starting time back into the past, show that this stationary process makes sense on all of $\mathbb{R}$. Show that if B_t is Brownian motion then $e^{-t/2}B_{e^t}$ is equal in distribution to this stationary Ornstein-Uhlenbeck process.

Exercise 1.8. Let $\mathcal{E} = [0,\infty)^2$ and ξ be white noise on $\mathcal{E}$. Define

$$B(t,s) = \int \mathbf{1}_{[0,t)}(x)\mathbf{1}_{[0,s)}(y)\xi(x,y)dxdy.$$

This is the *Brownian sheet*. Show that if we look along any line parellel to the axes we get Brownian motion with some diffusivity σ^2. Let $X(t,s) = e^{-t-s}B(e^{2s}, e^{2t})$. Show that if we look along any line parallel to the axis we get an Ornstein-Uhlenbeck process.

Continuing the discussion, our model equation will therefore be the noisy viscous Hamilton-Jacobi equation

$$\partial_t h = H(\partial_x h) + \partial_x^2 h + \xi \tag{15}$$

where ξ is space-time white noise. The simplest example is

$$H(\partial_x h) = (\partial_x h)^2 \tag{16}$$

In fact what is done in physics is to take a general H and expand

$$H(\partial_x h) = H(0) + H'(0)\partial_x h + \tfrac{1}{2}H''(0)(\partial_x h)^2 + \cdots \tag{17}$$

The first term can be removed from the equation by a time shift. The second should vanish by symmetry, but anyway could be removed from the equation by a constant velocity shift of coordinates. Thus the quadratic term is the first nontrivial contribution, and it is the only one kept. This is justified by claiming that the expansion is done with $\partial_x h$ small. We arrive at the KPZ equation,

$$\partial_t h = (\partial_x h)^2 + \partial_x^2 h + \xi. \tag{18}$$

There is something wrong with this derivation. The problem is that $\partial_x h$ is not small. As we will see later $|\partial_x h|^2$ is huge. So one needs to subtract a huge term reflecting the small scale fluctuations and the equation should really be written

$$\partial_t h = (\partial_x h)^2 - \infty + \partial_x^2 h + \xi. \tag{19}$$

Amazingly, through such a naive derivation, one finds a non-trivial field (see [ALB95b], [HHZ95], [ALB95a] for introductions.) Correct derivations are not easy: One can be found in [HQ18]. It is highly non-trivial, as the equality is one between distributional objects. In PDE one is used to weak solutions, but usually they are shown later to be at least piecewise regular, with some shocks. Here it is all shocks and there is a jump to a new type of equation representing natural phenomena where the equality is really between distributions, and some physical mechanism determines how to interpret non-linearities.

Formally, the KPZ equation is equivalent to the stochastic Burgers equation

$$\partial_t u = -\lambda\partial_x u^2 + \nu\partial_x^2 u + \sqrt{D}\partial_x \xi. \tag{20}$$

which, if things were nice, would be satisfied by $u = \partial_x h$. Note that for the stochastic Burgers equation the funny infinity goes away.

1.1. **1:2:3 Scaling.** Start with

$$\partial_t h = \lambda(\partial_x h)^2 + \nu\partial_x^2 h + \sqrt{D}\xi \tag{21}$$

Rescale

$$h_\epsilon(t,x) = \epsilon^b h(\epsilon^{-z}t, \epsilon^{-1}x) \tag{22}$$

and we have $\partial_t h = \epsilon^{z-b}\partial_t h_\epsilon$, $\partial_x h = \epsilon^{1-b}\partial_x h_\epsilon$ and $\partial_x^2 h = \epsilon^{2-b}\partial_x^2 h_\epsilon$. From (7) we have

$$\partial_t h_\epsilon = \epsilon^{2-z-b}\lambda(\partial_x h_\epsilon)^2 + \epsilon^{2-z}\nu\partial_x^2 h_\epsilon + \epsilon^{b-\frac{1}{2}z+\frac{1}{2}}\sqrt{D}\xi. \tag{23}$$

Clearly this means we can choose $\lambda = \nu = 1/2$ and $D = 1$ which is a kind of standard case. Also, when comparing discrete models to KPZ, one uses such arguments to identify the appropriate λ, ν and D. This is called *KPZ scaling theory* (see [Spo14] for a discussion.)

Most important though is that the scaling allows us to identify the non-trivial scale of fluctuations. It needs an extra piece of information, which is that Brownian motion is invariant for KPZ, except for a global height shift. This is not easy to prove, or even to understand. The best we have is that we can prove random walks are invariant for some discrete models which scale to the KPZ equation, so we know it is true. So KPZ is like a dynamical Brownian motion. It is not too hard to understand why z is called the *dynamical scaling exponent.* Its the new non-trivial thing we want to get our hands on. The invariance of Brownian motion plus the diffusive scaling of Brownian motion means that to see any non-trivial limit we must have

$$b = 1/2 \tag{24}$$

in (22). Then (23) reads

$$\partial_t h_\epsilon = \epsilon^{3/2-z}\lambda(\partial_x h_\epsilon)^2 + \epsilon^{2-z}\nu\partial_x^2 h_\epsilon + \epsilon^{1-\frac{1}{2}z}\sqrt{D}\xi. \tag{25}$$

If we had $z < 3/2$ everything on the right just goes to zero. If we had $z > 3/2$ the first term blows up. So we have to have [PR75],[FNS77]

$$z = 3/2. \tag{26}$$

Note that $b = 1/2$ says our fluctuations are supposed to be of size $\epsilon^{-1/2}$ at time $\epsilon^{-3/2}$ so this says that the fluctuations at large time t are of size $t^{1/3}$.

Exercise 1.9. Linearizing (18) one obtains the *Edwards-Wilkinson* equation,

$$\partial_t h = \tfrac{1}{2}\partial_x^2 h + \xi, \qquad h(0,x) = h_0(x) \tag{27}$$

whose solution is the infinite dimensional Ornstein-Uhlenbeck process,

$$h(t,x) = \int_{\mathbb{R}} p(t,x-y)h_0(y)dy + \int_0^t \int_{\mathbb{R}} p(t-s,x-y)\xi(s,y)dyds \tag{28}$$

where

$$p(t,x) = \frac{1}{\sqrt{2\pi t}}e^{-x^2/2t}. \tag{29}$$

Show that

$$E[(h(t,y)-h(t,x))^2] \sim C|y-x| \tag{30}$$

as y becomes close to x. Solve (27) with periodic boundary conditions by taking Fourier series, and show that a Brownian "loop" is invariant. Don't try to prove it, but this implies that Brownian motion is invariant for the model on the whole line. Now show that the Edwards-Wilkenson equation has 1:2:4 scaling, and in particular the fluctuations are $\mathcal{O}(t^{1/4})$. So the non-linear term makes these models *super-diffusive* as opposed to *sub-diffusive*. The $\mathcal{O}(t^{1/2})$ fluctuations mentioned before for the ballistic aggregation without the stickiness is because there is no spatial coupling at all and is completely non-physical. Compute the equilibrium space-time correlation functions

$$E[\partial_x h(t,x)\partial_x h(0,0)] \tag{31}$$

(i.e. assuming one starts with a Brownian motion.)

Finally, check from (28) that for fixed $t > 0$, $h(t,x)$ is locally Brownian.

KPZ can be written down in higher dimensions,

$$\partial_t h = |\nabla h|^2 + \Delta h + \xi \tag{32}$$

with $x \in \mathbb{R}^d$. It is not expected to make sense with space-time white noise forcing. But the dynamic scaling we just described only looks at large scales. So it would be the same if we had smoothed out the noise, or studied some discrete model which has the main features of KPZ (non-linearity, smoothing, forcing noise with no long range correlations). This is the idea behind universality. So in $d > 1$ we could just start with the smoothed out white noise, or some discrete model. The trouble is that no one knows what the invariant measure is, or even what it looks like. So one cannot get started on the scaling argument. In $d = 2$ very good computer simulations suggest that the fluctuations are of size $t^{0.24\cdots}$. "Very good" means that the physicist who do them trust them sufficiently that they are fairly sure the exponent is really not $1/4$.

If we look at (25) with dynamical exponent $z = 3/2$ we get

$$\partial_t h_\epsilon = \lambda(\partial_x h_\epsilon)^2 + \epsilon^{1/2}\nu\partial_x^2 h_\epsilon + \epsilon^{\frac{1}{4}}\sqrt{D}\xi. \tag{33}$$

So it is really some perturbation of

$$\partial_t h = \lambda(\partial_x h)^2. \tag{34}$$

Unfortunately, (34) is not well-posed.

Exercise 1.10. Set $\lambda = 1/2$ and differentiate to get the equivalent *inviscid Burgers' equation*

$$\partial_t u = u\partial_x u \tag{35}$$

Look for *characteristic curves* $x(t)$ along which the solution is constant $\partial_t u(t, x(t)) = 0$ to find $\dot{x} = -u(t, x)$. Now solve with $u(0, x) = -x/|x|$ and find many solutions.

Exercise 1.11 (Cole-Hopf transformation)**.** Let V be a nice function of t and x. Show that $h = \nu\lambda^{-1}\log z$ transforms the equation

$$\partial_t h = \lambda(\partial_x h)^2 + \nu\partial_x^2 h + V \tag{36}$$

into the equation

$$\partial_t z = \nu\partial_x^2 z + \lambda\nu^{-1}Vz. \tag{37}$$

Exercise 1.12. You may have heard of *viscosity solutions* of (34). You may even have been told these are the "true" solutions. There is an intrinsic notion of viscosity solutions, which we will not deal with, but the original idea was that one should add some viscosity, solve the equation, then remove it to find the physical solution. After all, there is supposed to be a tiny bit of friction around always. So we solve

$$\partial_t h_\epsilon = \lambda(\partial_x h_\epsilon)^2 + \epsilon^{1/2}\nu\partial_x^2 h_\epsilon \tag{38}$$

and let ϵ go to zero to find our solution. Use the Cole-Hopf transformation to find an explicit solution of (38) in terms of convolution of the initial condition with heat kernels. Now take a limit as $\epsilon \to 0$ to obtain the variational formula for the resulting limit

$$h(t, x) = \sup_y \left\{ h(0, y) - \tfrac{(x-y)^2}{4\lambda t} \right\}. \tag{39}$$

If you let $h(0, x) =$ two sided Brownian motion, you can actually compute the resulting $h(t, x)$. It is not easy, it can be found in [FM00] (following [Gro89]). The important point though is that one can easily check from the formula in [FM00] that $h(t, \cdot)$ is *not* a two-sided Brownian motion. Exercise: check that the resulting process is 1:2:3 invariant. It is an example of a 1:2:3 invariant process that is not the KPZ fixed point.

Since two-sided Brownian motion is invariant for (33) for every ϵ, we believe that it should be invariant for whatever the limit is. In other words, we can see the limit we are after is more subtle than just the viscosity solution. What happens is that a weird remnant of the noise survives in the limit. Identifying it is the main goal of the course.

Cole-Hopf does another thing for us. The KPZ equation as written makes no sense. As we saw in Exercise 1.3, the derivative of Brownian motion is white noise on $\mathbb{R}$. White noise is a distribution and the trouble with distributions is that you can only do linear things to them, i.e. their square is not unambiguously defined. Of course there are a lot of theories about how to define such things, but the key point is that there is not a unique canonical choice. One has to do it the right way to match the physics. In fact, there is a textbook

(I won't say its name, but it is green) where the KPZ equation is purportedly solved using a particular choice of the square. The resulting process was shown to not have the correct 1:2:3 scaling, so the theory in the book does not correspond to a physical theory of KPZ.

It turns out the following very simplistic approach does lead to the correct definition. Suppose the noise were nice. Then

$$h(t,x) = \log z(t,x) \tag{40}$$

turns

$$\partial_t z = \tfrac{1}{2}\partial_x^2 z + z\xi. \tag{41}$$

into KPZ

$$\partial_t h = \tfrac{1}{2}(\partial_x h)^2 + \tfrac{1}{2}\partial_x^2 h + \xi \tag{42}$$

Exercise 1.13. Take the noise to be white noise smoothed out by convolving with a smooth kernel in space of the form $\epsilon^{-1}\phi(\epsilon^{-1}x)$, but not in time. Develop an Itô formula in this context show that the non-linearity is replaced by $(\partial_x h)^2 - C\epsilon^{-1}$. Compute C.

2. Regularity of stochastic processes and random fields

There's an underlying probability space $(\Omega, \mathcal{F}, P)$ and a parameter set $\mathcal{E}$ and we have random variables X_t, $t \in \mathcal{E}$. For example, in the case of Brownian motion $\mathcal{E}$ might be $[0,\infty)$ and in the case of KPZ, $\mathcal{E}$ would be $[0,\infty)\times\mathbb{R}$. An even worse example is white noise where $\mathcal{E}$ is a set of test functions. Usually we describe the thing through the finite dimensional distributions, which means if $F \subset \mathcal{E}$ is finite then the random vector X_t, $t \in F$ has a probability distribution μ_F. They have to be consistent in the sense that if $F_1 \subset F_2$ then μ_{F_1} is the projection of μ_{F_2}. Then Kolmogorov has a theorem which is kind of a version of the Caratheodory extension theorem that there is a unique measure on the product space with these finite dimensional distributions. The trouble is that the measure is on the product $\sigma-$field, which does not even have enough information to tell you the resulting function is continuous (i.e. the set of continuous functions would not be measurable, since the product topology really only knows about finitely many coordinates). On the other hand, on a countable $\mathcal{E}$, this is enough. The real problem is that in probability everything is up to sets of measure zero and if $\mathcal{E}$ is uncountable we have to face the horrible issue of whether an uncountable union of sets of measure zero may add up to something non-trivial. This is why regularity of these fields is an important issue. If we know that the process, say Brownian motion, is continuous, then it is definted by its values on a dense countable set. The finite dimensional distribution of Brownian motion are for $0 < t_1 < \cdots < t_n$

$$P(B_{t_1} \in A_1, \ldots, B_{t_n} \in A_n) = \int_{A_n} \cdots \int_{A_1} \frac{e^{-\frac{x_1^2}{2t_1}}}{\sqrt{2\pi t_1}} \cdots \frac{e^{-\frac{(x_n - x_{n-1})^2}{2(t_n - t_{n-1})}}}{\sqrt{2\pi(t_n - t_{n-1})}} dx_1 \cdots dx_n \tag{43}$$

Theorem 2.1 (Wiener)**.** *There is a unique measure P on the space $C([0,\infty))$ of continuous functions with the finite dimensional distributions* (43)*.*

Proof. We construct it first on $C([0,1])$. Then one can construct an independent copy on each $C([n,n+1])$ and concatenate them so the result is continuous and has the right finite dimensional distributions.

To construct it on $C([0,1])$ we will use the dyadics $j/2^n$ which are a nice dense set $D \subset [0,1]$. The dyadics at level n are $j/2^n$, $j = 0,1,\ldots,2^n$ which we call D_n. On D_n we can just use a sequence of independent Gaussians to construct $B_{j/2^n}$. Then we fill it in with polygonal lines to make a continuous function $B^{(n)}_t$, $t \in [0,1]$. Clearly, when n is large, this is a good approximation of the Brownian motion. We do this for every n at once and call the background probability measure Prob to distinguish it from the measure P we are trying to create on $C([0,1])$. Now, if we can show that

$$\mathrm{Prob}\left(\sup_{0\le t\le 1} |B^{(n+1)}_t - B^{(n)}_t| \ge 2^{-\gamma n}\right) \le C2^{-\delta n} \tag{44}$$

then we can conclude by Borel-Cantelli that

$$\mathrm{Prob}\left(\lim_{n\to\infty} B^{(n)}_t = B_t \text{ exists uniformly on } [0,1]\right) = 1 \tag{45}$$

and our problem is solved. Now notice that because of the dyadic lattices, the sup in (44) is achieved at one of the midpoints of D_n, in particular it is less than or equal to

$$\sup_{1\le j\le 2^n} \max\left\{|B^{(n+1)}_{\frac{2j-1}{2^{n+1}}} - B^{(n+1)}_{\frac{2j}{2^{n+1}}}|, |B^{(n+1)}_{\frac{2j-1}{2^{n+1}}} - B^{(n+1)}_{\frac{2j-2}{2^{n+1}}}|\right\} \tag{46}$$

For the $\sup_{1\le j\le 2^n}$ and the max we just use the union bound, so that the probability on the left hand side of (44) can be bounded by

$$2^{n+1}P(|B_{t+1/2^n} - B_t| \ge 2^{-\gamma n}). \tag{47}$$

Note that we can think of this as the Brownian probabilities so I just wrote P instead of Prob, and dropped the superscript. We could just compute this, but let's do something which will give us a more general result below. Take $\beta > 0$ and estimate by Chebyshev

$$P(|B_{t+1/2^n} - B_t| \ge 2^{-\gamma n}) \le 2^{-\beta\gamma n} E[|B_{t+1/2^n} - B_t|^\beta] \tag{48}$$

Define α by

$$E[|B_{t+1/2^n} - B_t|^\beta] \le C2^{n(1+\alpha)} \tag{49}$$

for all n. If $\alpha > 0$ then we can choose γ small enough that $\alpha - \beta\gamma = \delta > 0$ and we have proved (44). For Brownian motion we can choose $\beta = 4$ and then $\alpha = 1$ and we are done. □

The amazing thing about this proof is that it only used the two point function, i.e. one can deduce regularity from just the two dimensional marginals of a stochastic process.

Exercise 2.2 (Kolmogorov regularity theorem)**.** Suppose that the two dimensional marginals $\mu_{s,t}$ satisfy for some $\alpha, \beta > 0$ and $C < \infty$,

$$\sup_{0\le s<t\le T} \int\int |x-y|^\beta \mu_{s,t}(dx,dy) \le C|t-s|^{1+\alpha}. \tag{50}$$

Adapt the previous proof to show that there is a measure P on $C([0,T])$ with finite dimensional distributions μ_F.

Exercise 2.3 (Hölder continuity)**.** 1. Adapt the previous proof to show that Brownian motion is almost surely α-Hölder for any $\alpha < 1/2$. 2. Prove that it is not 1/2-Hölder by showing that

$$P\left(\sup_n \max_{0\le j\le 2^n-1} 2^{n/2}|B(\frac{j+1}{2^n}) - B(\frac{j}{2^n})| = \infty\right) = 1 \tag{51}$$

3. SDEs and SPDEs

Cole-Hopf tells us that if we can solve

$$\partial_t z = \tfrac{1}{2}\partial_x^2 z + z\xi, \tag{52}$$

then we can think of $h(t,x) = \log z(t,x)$ as our solution of KPZ. It is called the *Cole-Hopf solution of KPZ.* It is far from clear though, that (52) is any easier to make sense of than KPZ. But actually, it is a lot better. For fixed t, the solution $h(t,x)$ of KPZ looks locally like a Brownian motion in x. So $(\partial_x h)^2$ is like squaring a white noise. This looks really hard to make sense of. In the stochastic heat equation (81) the only problem is $z\xi$. Now $z = e^h$ is also locally Brownian, but it is multiplying the white noise ξ. So in KPZ it is like squaring a white noise and in the multiplicative stochastic heat equation it is like multiplying something locally Brownian times a white noise. It is still not easy though, and there is also the issue that it is a space-time white noise. To understand the difficulty note that by variation of constants the equation can be written in what is called *mild form*:

$$z(t,x) = \int p(t,x-y)z_0(y)dy + \int_0^t \int p(t-s,x-y)z(s,y)\xi(s,y)dyds. \tag{53}$$

Note here we've been a little casual about writing the white noise as if it were a function. We will do this as long as there is an integration involved and just think of it as a notation for the action of the random distribution against whatever it is being "integrated" against. At any rate, we have to face that we are integrating a pretty irregular object against another. To see that there is a real issue, even without the space-time, let's just recall the basics of stochastic integration in $d = 1$.

Exercise 3.1. Use the Borel-Cantelli lemma to show that the quadratic variation of Brownian motion

$$\lim_{n\to\infty} \sum_{j=0}^{\lfloor 2^n t\rfloor -1} |B_{\frac{j+1}{2^n}} - B_{\frac{j}{2^n}}|^2 \tag{54}$$

converges to t almost surely.

Let's try to define the stochastic integral

$$\int_0^t B(s)dB(s). \tag{55}$$

Two simple options to approximate this are

$$L_n = \sum_{j=0}^{\lfloor 2^n t\rfloor -1} B_{\frac{j}{2^n}}(B_{\frac{j+1}{2^n}} - B_{\frac{j}{2^n}}) \qquad R_n = \sum_{j=0}^{\lfloor 2^n t\rfloor -1} B_{\frac{j+1}{2^n}}(B_{\frac{j+1}{2^n}} - B_{\frac{j}{2^n}}) \tag{56}$$

If we take $L_n + R_n$ it is an alternating sum and the limit is easily seen to be B_t^2. If we take $L_n - R_n$ on the other hand, and this is the key point, it is just the approximate quadratic variation in (54) so it goes to t instead of 0. Now we can see that even a seemingly easier problem than the $(\partial_x h)^2$ in KPZ leads to the right hand side of (53) which has clearly got the same kind of problem. In fact, since white noise in one dimension is the derivative of Brownian motion the integral in (55) is really trying to integrate $B\xi$ which is really very analogous.

The Itô integral means the choice of left endpoint, with the distinct advantage that the result is a martingale. This is formalized by considering only predictable[1] functions $f(t,\omega)$ as integrands and we have the Itô isometry

$$E\left[\left(\int f dB\right)^2\right] = E\left[\int f^2 dt\right] \tag{57}$$

3.1. **Wiener chaos.** Note that the issue above does not come up if we are computing $\int f dB = \langle f, \xi\rangle$ when $f = f(t)$ is non-random and in $L^2[0,\infty)$. Then define recursively for $f_n \in L^2(\Delta_n)$ where $\Delta_n = \{0 \le t_1 \le \cdots \le t_n \le T\}$,

$$I_n(f_n) = \int_{\Delta_n} f_n(t_1,\ldots,t_n) dB_{t_1}\cdots dB_{t_n} \tag{58}$$

and $\mathcal{H}_n$ to be the Hilbert subspace of $L^2(\Omega,\mathcal{F},P)$ spanned by $I_n(f_n),\ f_n \in L^2(\Delta_n)$. The next exercise shows that this actually captures everything.

Exercise 3.2. Use the Itô isometry to show that the $\mathcal{H}_n$ are orthogonal, i.e.

$$E[I_n(f_n)I_m(f_m)] = \delta_{n,m}\|f_n\|^2_{L^2(\Delta_n)} \tag{59}$$

Exercise 3.3. If $f \in L^2([0,T])$ call $f^{\otimes n}(t_1,\ldots,t_n) = f(t_1)\cdots f(t_n)$. The linear span of such functions is dense in $L^2(\Delta_n)$. The linear span of $I_n(f^{\otimes n})$ is dense in $\mathcal{H}_n$. The Hermite polynomials

$$H_n(x) = (-1)^n \frac{1}{n!} e^{\frac{x^2}{2}} \frac{d^n}{dx^n} e^{-\frac{x^2}{2}} \tag{60}$$

can also be defined through

$$e^{tx-\frac{1}{2}t^2} = \sum_{k=0}^{\infty} t^k H_k(x). \tag{61}$$

Hence if

$$M_\lambda(t) = e^{\lambda \int_0^t f dB - \frac{\lambda^2}{2}\int_0^t f^2 ds} \tag{62}$$

then

$$M_\lambda(T) = \sum_{k=0}^{\infty} \lambda^k \|f\|^k H_k(I_1(f)/\|f\|). \tag{63}$$

Next prove that

$$M_\lambda(t) = 1 + \lambda \int_0^t f(s) M_\lambda(s) dB_s. \tag{64}$$

Iterate to get

$$M_\lambda(T) = 1 + \sum_{k=1}^{\infty} \lambda^k I_k(f^{\otimes k}). \tag{65}$$

(You actually have to check here that the term left over when you iterate n times goes to zero.) Conclude that

$$I_n(f^{\otimes n}) = \|f\|^n H_n\left(\frac{1}{\|f\|}\int_0^T f dB\right). \tag{66}$$

[1] i.e. the closure in $L^2([0,\infty)\times\Omega)$ of simple functions $\sum_{i=1}^n f_i(\omega)\mathbf{1}_{[t_i,t_{i+1})}(t)$ with $f_i \in \mathcal{F}_{t_i}$ where $\mathcal{F}_t = \sigma(B_s, s \le t)$.

If $F \in L^2(\Omega, \mathcal{F}_T, P)$ and $E[FI_n(f_n)] = 0$ for every n and every $f_n \in L^2(\Delta_n)$ conclude that $E[FM_\lambda(T)] = 0$ for every λ and therefore that $F = 0$[2]. Conclude finally that

$$L^2(\Omega, \mathcal{F}_T, P) = \bigoplus_{k=0}^{\infty} \mathcal{H}_k \simeq \bigoplus_{k=0}^{\infty} L^2(\Delta_k). \tag{67}$$

$\mathcal{H}_k$ is called the $k-$th chaos. The direct sum of $L^2(\Delta_k)$ is known as *Fock space.*

Exercise 3.4. Use the previous exercise to solve the equation

$$dx = axdt + bxdB \tag{68}$$

as a chaos series.

3.2. **Feynman-Kac.** There's one more thing we need from basic stochastic calculus. We know that if B_t is Brownian motion then

$$u(t,x) = E_x\left[f(B_t)\right] \tag{69}$$

where the E_x indicates that $B_0 = x$, solves the heat equation

$$\partial_t u = \tfrac{1}{2}\partial_x^2 u, \qquad u(0,x) = f(x). \tag{70}$$

There is nothing much to prove here because the left hand side is by definition of the Brownian one-dimensional marginals

$$\int f(y) \frac{e^{-\frac{(x-y)^2}{2t}}}{\sqrt{2\pi t}} dy \tag{71}$$

which is the solution[3] and that is that. Note that we can write this as

$$(e^{\frac{t}{2}\Delta} f)(x) = E_x\left[f(B_t)\right]. \tag{72}$$

You can check it works the same if B_t is a $d-$dimensional Brownian motion.

One day Richard Feynman was explaining the path integral approach to quantum mechanics, Marc Kac[4] was in the audience and pointed out that if one took the i out of the exponent (called "Euclidean shift") he could make sense of what Feynman was saying. No one has ever been able to make the original approach rigorous, but it has been incredibly influential. The rigorous Feynman-Kac formula is also super-useful.

Theorem 3.5 (Feynman-Kac formula)**.** *As long as $V(t,x)$ is nice (in the simple proof below we will assume f, V is bounded),*

$$u(t,x) = E_x\left[e^{\int_0^t V(t-s,B_s)ds} f(B_t)\right] \tag{73}$$

is a representation using Brownian motion of the solution of

$$\partial_t u = \tfrac{1}{2}\Delta u + V(t,x)u, \qquad u(0,x) = f(x). \tag{74}$$

[2]Hint: To prove this note first that it is enough to show that F is orthogonal to cylinder functions. Then note that by Fourier inversion $E[F\phi(B_{t_1}, B_{t_2} - B_{t_1}, \ldots, B_{t_n} - B_{t_{n-1}})] = \int \hat{\phi}(z_1, \ldots, z_n) E[F e^{2\pi i \sum z_j (B_{t_{j+1}} - B_{t_j})}] dz_1 \cdots dz_n$

[3]One has uniqueness on $[0,T]$ basically as long as (71) is absolutely integrable up to T. Otherwise there are counterexamples. See the book by Widder.

[4]pronounced "Cats"

Nice here basically means that (74) is well-posed. Books have been written on what V are acceptable. If $V(t,x) = V(x)$ doesn't depend on t then we are basically asking if $H = \frac{1}{2}\Delta + V$ is essentially self-adjoint and then we can write

$$e^{tH} f(x) = E_x \left[e^{\int_0^t V(B_s)ds} f(B_t) \right]. \tag{75}$$

For details, read all four volumes of Reed & Simon plus Simon's "Functional integration and quantum physics". Note that if $V(t,x)$ is bounded above everything is definitely ok though you don't see that in the proof below. The formula is written in many equivalent ways in different textbooks. For example, it is often natural to reverse the time in the integral so we would have $\int_0^t V(s, B_{t-s})ds$. But then one can think of a Brownian motion coming in reverse time from x at time t and maybe write it as

$$E^x \left[e^{\int_0^t V(s,B_s)ds} f(B_0) \right]. \tag{76}$$

Here is an elementary proof of the Feynman-Kac formula.

Proof. By variation of constants formula

$$u(t,x) = \int p(t, x-y) f(y) dy + \int_0^t \int p(t-s, x-y) u(s,y) V(s,y) dy ds \tag{77}$$

If we use this formula in the last expression, then do it again, i.e. apply this recursively, we get

$$\sum_{k=0}^{\infty} \int_{0 \le t_1 \le \cdots \le t_k \le t} \int \cdots \int \prod_{i=1}^{k} p(t_{i+1} - t_i, y_{i+1} - y_i) V(t_i, y_i) p(t_1, y_1 - y_0) f(y_0) dt_i dy_i \tag{78}$$

If we expand the exponential in (76) and eat the factorial by restricting to the Weyl chamber, we get by Fubini

$$\sum_{k=0}^{\infty} \int_{0 \le t_1 \le \cdots \le t_k \le t} E^x \left[\prod_{i=1}^{k} V(t_i, y_i) f(B_0) \right] \prod_{i=0}^{k} dt_i \tag{79}$$

which is just another way of writing the same thing.

There is a gap here – there had to be, otherwise it would always be true – that we can really only do the recursion up to a finite n and then we have to show the error goes to zero. If $f, V \le C$ then it is easy to bound $|u(t,x)| \le Ce^{Ct} \le C'$ from the equation (74). Then the nth term in the recursion is less than or equal in absolute value to $\frac{C'}{n!} E^x[(\int_0^t V(s,B_s)ds)^2]$. But if $|V| \le C$ this is clearly summable. □

3.3. **Multiplicative stochastic heat equation.** We're still trying to define what it means to be a solution of (81). The general idea of what we mean by a solution of an SPDE is that we smooth out the noise, let ϱ be a compactly supported, non-negative, smooth, even function on $\mathbb{Z}^2$ with total integral 1 and $\varrho_\epsilon(t,x) = \epsilon^{-1}\varrho(\epsilon^{-1/2}t, \epsilon^{-1/2}x)$ our approximate identity and

$$\xi_\epsilon = \varrho_\epsilon * \xi, \tag{80}$$

solve

$$\partial_t z_\epsilon = \tfrac{1}{2}\partial_x^2 z_\epsilon + z_\epsilon \xi_\epsilon \tag{81}$$

hopefully in some classical way, and try to take a limit as $\epsilon \to 0$. If there is a limit which doesn't depend too badly on the choice of mollifier, then we say that is the solution. In our case, if there is enough mollification to get a reasonable function ξ_ϵ we actually have a nice representation of the solution by the Feynman-Kac formula,

$$z_\epsilon(t,x) = E^x\left[e^{\int_0^t \xi_\epsilon(s,B_s)ds} z(0,B_0)\right]. \tag{82}$$

As we let $\epsilon \to 0$ all hell breaks loose as the Brownian motion tries to find places where the potential is huge in order to increase the exponent.

We already saw that in the one dimensional case of SDE's the solution depends on the way we mollify. There was a constant which gave us the difference between using the Itô and backwards Itô prescriptions. We can always keep track of these constants and make a simple rule to go back and forth. But in the SPDE case we face the problem that these constants are often infinite. It is like you take a step sideways and follow a path along the side of the hill that runs pretty much parallel and gets you to essentially the same place. But when the dimension becomes large, as we go to SPDE's, the hill becomes steeper and steeper until the paths are still parallel, but infinitely far from each other. If you do the convolution in (80) in time, you definitely get big problems like this. It is better to just convolve the noise in space and let it stay white in time. If you do this you can make a theory that isn't all that different from the finite dimensional Itô theory. Since it is not the main point of this class, we'll do something morally equivalent, which is to discretize space. So we replace (81) by

$$dz_{i/2^n} = \tfrac{1}{2}(z_{i+1/2^n} - 2z_{i/2^n} + z_{i-1/2^n})dt + \sigma_n z_{i/2^n} dB_{i/2^n} \tag{83}$$

where the $B_{i/2^n}(t)$ are independent Brownian motions[5]. We have to determine σ_n. To have it properly approximate the space-time white noise we need

$$E[(\int \frac{1}{2^n}\sum_{i=-\infty}^{\infty} f(i/2^n,s)\sigma_n dB_{i/2^n})^2] = \int \frac{1}{2^n}\sum_{i=-\infty}^{\infty} f^2(i/2^n,s)ds \tag{84}$$

which says that $\sigma_n = 2^{n/2}$. This means that every time you use Itô's formula you pick up a factor 2^n which basically means when you apply Itô with SPDE's the Itô corrections are all infinite. In fact, if z is the solution of

$$\partial_t z = \tfrac{1}{2}\partial_x^2 z + z\xi, \tag{85}$$

then by Itô's formula $h(t,x) = \log z(t,x)$ solves

$$\partial_t h = \tfrac{1}{2}(\partial_x h)^2 - \infty + \tfrac{1}{2}\partial_x^2 h + \xi \tag{86}$$

which is really, honest-to-goodness, what KPZ means.

Exercise 3.6. Let $p^{(n)}(t,x)$, $x \in 2^{-n}\mathbb{Z}$ be the kernel of

$$dz_{i/2^n} = \tfrac{1}{2}(z_{i+1/2^n} - 2z_{i/2^n} + z_{i-1/2^n})dt. \tag{87}$$

Show that a solution of (83) with $z(0,y) = 2^n \mathbf{1}_{y=0}$ is given by the element whose kth chaos is represented by the function

$$C_n p^{(n)}(t-t_k, x-x_k)p^{(n)}(t_k-t_{k-1}, x_k-x_{k-1})\cdots p^{(n)}(t_2-t_1, x_2-x_1)p^{(n)}(t_1,x_1) \tag{88}$$

Exercise 3.7. Show that

$$p_t(j) = \frac{e^{-2t}}{2\pi i}\int_{\gamma_0} e^{t(z+z^{-1})}\frac{dz}{z^{j+1}} \tag{89}$$

[5]I know it is annoying that we change notations from $f_t(x)$ to $f_x(t)$ to $f(t,x)$ from line to line. Sorry.

where γ_0 is a simple contour around 0 is a representation of the fundamental solution of the discrete heat equation on $\mathbb{Z}$. Scale it to obtain the $p^{(n)}(t,x)$ from the previous exercise and take a limit $n \to \infty$ to recover the standard heat kernel.

Taking limits as $n \to \infty$ we obtain the chaos representation for the fundamental solution

$$z(t,x) = \sum_{k=0}^{\infty} \int_{\Delta_k} \int_{\mathbb{R}^k} \mathbf{p}_k(\mathbf{s}, y_0, \mathbf{y}) \xi^{\otimes k}(\mathbf{s}, \mathbf{y}) d\mathbf{y} d\mathbf{s}. \tag{90}$$

where $\Delta_k = \{0 \le s_1 < \cdots < s_k \le t\}$ and for such an $\mathbf{s} = (s_1, \ldots, s_k)$

$$\mathbf{p}_k(\mathbf{s}, y_0, \mathbf{y}) = p(t - s_k, x - y_k) p(s_k - s_{k-1}, y_k - y_{k-1}) \cdots p(s_2 - s_1, y_2 - y_1) p(s_1, y_1) \tag{91}$$

are transition densities for a Brownian bridge to be at y_i at times s_i, $i = 1, \ldots, k$ and end up at x at time t, given that the Brownian motion started at 0 at time 0.

Exercise 3.8. Show that,

$$\int_{\Delta_k} \int_{\mathbb{R}^k} \mathbf{p}_k^2(\mathbf{s}, y_0, \mathbf{y}) d\mathbf{y} d\mathbf{s} \le C(n!)^{-1/2} \tag{92}$$

so that the chaos series converges.

Exercise 3.9. Use the same ideas to "solve" the SDE $dX = XdB$.

3.4. **Duality.** Some SPDE's (and SDE's) can be "solved" because they happen to have a dual. We say that two processes x_t, Y_t on $[0,T]$ are in duality if there is a function $H(X,Y)$ such that for $t \in [0,T]$

$$E[H(X_t, Y_0)] = E[H(X_0, Y_t)] \tag{93}$$

In some cases, the Y_t may even be deterministic.

Exercise 3.10 (Special SDEs/SPDEs "solvable" by duality)**.**
1. (Feller diffusion) Show that $dX = \sqrt{X} dB$ is in duality with the ODE $\dot{v} = -\frac{1}{2} v^2$ through $H(X,v) = e^{-xv}$. Hint: Show that $e^{-X_t v(T-t)}$ is a martingale. Although the SDE does not have Lipschitz coefficients, it is well-posed by a method of Yamada-Watanabe.
2. (Dawson-Watanabe process, a.k.a. Super-Brownian motion) Show that $du = \frac{1}{2}\partial_x^2 u + \sqrt{u}\xi$ is in duality with $\partial_t v = \frac{1}{2}\partial_x^2 v - \frac{1}{2} v^2$ through $H(u,v) = e^{-\int v(x) u(x) dx}$. Hint: Martingale, use a discretization of space and take a limit. Don't worry about well-posedness of the SPDE. It is hard and the best that is known is uniqueness in distribution (which follows from this exercise. Can you prove it?)
3. (Fisher-Wright model) Show that $dX = \sqrt{X(1-X)} dB$ is in duality with the integer valued process N_t which jumps $N \to N-1$ at rate $\binom{N}{2}$ through $H(X,N) = X^N$.
4. Show that $du = \frac{1}{2}\partial_x^2 u + \sqrt{u(1-u)}\xi$ is in duality with a collection of Brownian motions which coalesce at rate 1 when they are together (i.e. for each pair i,j choose an exponential $\tau_{i,j}$ of rate 1 independent of everything and then decree that the paths of $B_i(t)$ and $B_j(t)$ coincide when their mutual local time $\ell_{i,j}(t) = \int_0^t \delta_0(B_i(s) - B_j(s)) ds$ reaches $\tau_{i,j}$. Call $\vec{B}(t)$ the Brownians at time t and N_t the number of them. The duality function is

$$H(u, \vec{B}) = \prod_{i=1}^{N} (1 - u(B_i)) \tag{94}$$

Hint: As before, discretize space and use a lattice approximation to the SPDE and approximate the Brownian motions with random walks. Prove the duality there and take a limit.

(5) (KPP equation) Show that the PDE $\partial_t u = \frac{1}{2}\partial_x^2 u + u(1-u)$ is in duality with branching Brownian motions with the duality function (94). Branching Brownian motions are just a collection of Brownian motions which each, at rate 1 split into two particles (i.e. they are independent Brownian motions after the splitting time). KPP is a model for traveling fronts. In the late 30's it was shown by Fisher[6] and Kolmogorov-Petrovski-Piscunov that if you start with a *Heaviside function*[7] $u_0(x) = \mathbf{1}_{x<0}$ then $v(t) = \inf\{x : u(t,x) \leq 1/2$ has $v(t) \simeq \sqrt{2}t$. In the late 70's Bramson famously showed using the duality with BBM that to next order $v(t) \simeq \sqrt{2}t - \frac{3}{\sqrt{2}}\log t$ which took over 20 years to reproduce without probability.

(6) (Stochastic Fisher-KPP) Show that the SPDE $\partial_t u = \frac{1}{2}\partial_x^2 u + u(1-u) + \lambda\sqrt{u(1-u)}\xi$ is in duality with branching, annihilating Brownian motions with the same duality function (94). Brunet-Derrida famously predicted that for small λ the speed is $\lim_{t\to\infty} v(t)/t = \sqrt{2}(1 - \frac{\pi^2}{(\log \lambda^2)^2})$, i.e. noise produces an unbelievably strong slowdown of the speed of the traveling front.

3.5. **Mild solutions of the multiplicative stochastic heat equation.** To make sense of the equation, we rewrite it in Duhamel form

$$z(t,x) = \int_{\mathbb{R}} p(t,x-y)z_0(y)dy - \int_0^t \int_{\mathbb{R}} p(t-s,x-y)z(s,y)\xi(s,y)dyds \tag{95}$$

using the kernel of the heat equation

$$p(t,x) = \frac{1}{\sqrt{2\pi t}}e^{-x^2/2t}. \tag{96}$$

If $z(t,x)$ is predictable[8] with

$$\int_0^t \int_{\mathbb{R}} p^2(t-s,x-y)E[z^2(s,y)]dyds < \infty \tag{97}$$

then the stochastic integral in (95) makes sense as an element of $L^2(P)$ as shown in the previous section. Such a $z(t,x)$ for which equality holds in (95) for all $0 \leq t \leq T$ and $x \in \mathbb{R}$ will be called a *mild solution* of the stochastic heat equation (**??**).

Of course we need to know such a solution exists. There are many ways to do it, but an easy way is Picard iteration: Let $z^0(t,x) = 0$ and

$$z^{n+1}(t,x) = \int_{\mathbb{R}} p(t,x-y)z_0(y)dy - \int_0^t \int_{\mathbb{R}} p(t-s,x-y)z^n(s,y)\xi(s,y)dyds. \tag{98}$$

They are progressively measurable by construction. Let $\bar{z}^n(t,x) = z^{n+1}(t,x) - z^n(t,x)$. Then

$$\bar{z}^{n+1}(t,x) = -\int_0^t \int_{\mathbb{R}} p(t-s,x-y)\bar{z}^n(s,y)\xi(s,y)dyds. \tag{99}$$

[6]Fisher made the mistake of publishing in the *Annals of Eugenics*

[7]Oliver Heaviside (1850 - 1925)

[8]In this 1+1 dimensional situation, this means that $\int z(t,x)\varphi(x)dx$ is predictable for any smooth compactly supported φ. The $\sigma-$field $\mathscr{F}_t$ is the one generated by $\langle\varphi,\xi\rangle$ for φ smooth compactly supported in $[0,t]\times\mathbb{R}$.

So

$$E[|\bar{z}^{n+1}(t,x)|^2] = \int_0^t \int_{\mathbb{R}} p^2(t-s,x-y)E[|\bar{z}^n(s,y)|^2]dyds. \tag{100}$$

Let $f^n(t) = \sup_{x,s\in[0,t]} E[|\bar{z}^n(s,x)|^2]$. Suppose that $f^0(t) < \infty$. Then we have, after integrating the heat kernel,

$$f^{n+1}(t) \leq C\int_0^t \frac{f^n(s)}{\sqrt{t-s}}ds. \tag{101}$$

Iterating the inequality,

$$f^{n+1}(t) \leq C\int_0^t \int_0^s \frac{f^{n-1}(u)}{\sqrt{(t-s)(s-u)}}duds. \tag{102}$$

Changing the order of integration

$$f^{n+1}(t) \leq C\int_0^t \int_u^t \frac{f^{n-1}(u)}{\sqrt{(t-s)(s-u)}}dsdu = C'\int_0^t f^{n-1}(u)du. \tag{103}$$

Now Gronwall's inequality implies that $f^n(t) \leq (C't)^{n/2}/(n/2)!$, so there is a limit $z(t,x)$ which is progressively measurable and solves the equation.

If we examine what $f^0(t) < \infty$ means, it is just that $\sup_x E[z_0^2(x)] < \infty$. so we obtain the following existence and uniqueness result:

Theorem 3.11 (Existence and uniqueness of mild solutions)**.** *Suppose that z_0 satisfies (??). Then there exists a unique progressively measurable $z(t,x)$ satisfying (95) and (97).*

3.6. **Directed polymers in random environment.** The chaos series (90) provides a rigorous version of the formula

$$E_0^x[e^{-\beta\int_0^t \xi(s,B_s)ds}] \tag{104}$$

with a Brownian bridge coming down from x at time t to 0 at time 0. What happened is that we expanded the exponential and removed the diagonals in a way embedded in the definition of the chaoses the made the whole thing make sense. Physicists write it as

$$\int_{\text{paths}} e^{-\beta\int_0^t \xi(s,B_s)ds - \frac{1}{2}\int_0^t |\dot{B}_s|^2 ds} \tag{105}$$

and think of it as an elastic band in a random background. There are several discrete versions of the model where you don't have to worry about whether the thing makes sense. For example, we can take a lattice version where X_k is a random walk and $\xi(i,j)$, $i = 1,2,\ldots$, $j \in \mathbb{Z}$ are i.i.d. and look at

$$Z(n,x) = E_0^x\left[e^{-\beta\sum_{k=1}^n \xi(k,X_k)}\right]. \tag{106}$$

They are all supposed to behave the same on a large scale, i.e. after subtraction of an appropriate constant c_n the 1:2:3 rescaled logarithm of z should converge to the KPZ fixed point (to be described). Nobody can prove it except for scattered results for a few special solvable models. We describe a few now.

3.7. **Special directed polymer in random environment models.**

(1) (Log-gamma polymer) $\beta = 1$ and $e^{\xi}(j,k) \sim \text{Gamma}(\mu, 1)$ i.e. has density $\frac{1}{\Gamma(\mu)} x^{\mu-1} e^{-x} \mathbf{1}_{x>0}$. Seppalainen discovered the amazing fact that if we look at walks on $\mathbb{Z}_+^2$ which at each step choose to go horizontally or vertically, and we choose the ξ on the horizontal boundary $(n, 0)$ to be $\text{Gamma}(\mu - \theta, 1)$ and on the vertical boundary $(0, n)$ to be $\text{Gamma}(\theta, 1)$,then for any *downright path* we have horizontal step ratios $Z(n+1, m)/Z(n, m) \sim \text{Gamma}(\mu - \theta, 1)$ and verticle step ratios $Z(n, m+1)/Z(n, m) \sim \text{Gamma}(\theta, 1)$, i.e. it has an explicit invariant measure. An formula for the distribution of the partition function is given in [Cor+14].
(2) (O'Connell-Yor (Brownian) polymer)
(3) (Geometric and exponential last passage)

3.8. **Moments of the multiplicative stochastic heat equation via replicas, delta-Bose gas and the narrow wedge solution of KPZ.** Suppose we solve the multiplicative stochastic heat equation (81) with the smoothed out the noise as in (80). This time lets use a smoothing kernel that smooths in both space and time. Then ξ_ϵ is a perfectly nice function and we can write the solution using the Feynman-Kac formula

$$z_\epsilon(t, x) = E^x \left[e^{\int_0^t \xi_\epsilon(s, B_s) ds - C_\epsilon t} z(0, B_0) \right] \tag{107}$$

We can figure out what C_ϵ is because if we look at our mild solution, it is supposed to be a martingale. Using $\mathbb{E}$ to denote expectation over the forcing white noise we must have $\mathbb{E}[E^x[e^{\int_0^t \xi_\epsilon(s,B_s)ds - C_\epsilon t}]] = 1$. If we want to compute it we can use Fubini (integrand is non-negative),

$$\mathbb{E}\left[E^x \left[e^{\int_0^t \xi_\epsilon(s,B_s)ds} z(0, B_0) \right] \right] = E^x \left[\mathbb{E}\left[e^{\sum_{i=1}^n \int_0^t \xi_\epsilon(s,B_s)ds} \right] \right] \tag{108}$$

Now write

$$\int_0^t \xi_\epsilon(s, B_s) ds = \int \xi f_\epsilon dt dx \tag{109}$$

where f_ϵ is the density of the convolution by ϱ_ϵ of the measure $\int_0^t \delta_{s,B_s} ds$. This is clearly $N(0, \|f_\epsilon\|_2^2)$ and therefore $\mathbb{E}[e^{\int_0^t \xi_\epsilon(s,B_s)ds}] = \mathbb{E}[e^{\frac{1}{2}\|f_\epsilon\|_2^2}]$. Obviously if we try to take $\epsilon \to 0$ this diverges and the divergence depends on the smoothing kernel. Let's leave it for a second because what we really want to compute is

$$\mathbb{E}[z(t, x_1) \cdots z(t, x_N)] \tag{110}$$

By the same argument

$$\mathbb{E}\left[\prod_{i=1}^N E^{x_i} \left[e^{\int_0^t \xi_\epsilon(s,B_s)ds} z(0, B_0) \right] \right] = E^{x_1,\dots,x_N} \left[\mathbb{E}\left[e^{\sum_{i=1}^n \int_0^t \xi_\epsilon(s,B_s^i)ds} \right] \prod_{i=1}^n z(0, B_0^i) \right] \tag{111}$$

where $E^{x_1,\dots,x_N}$ is the expectation with respect to N independent copies ("replicas") of the Brownian motion, ending at $x_1, \dots, x_N$. As above we have

$$\mathbb{E}[z(t, x_1) \cdots z(t, x_N)] = \lim_{\epsilon \to 0} E^{x_1,\dots,x_N} [e^{\frac{1}{2}(\|\sum_{i=1}^N f_\epsilon^i\|_2^2 - \sum_{i=1}^N \|f_\epsilon^i\|_2^2}] \tag{112}$$

where f_ϵ^i refers to the convoluted occupation measure of the ith Brownian motion. It is not so hard to see that the limit in the exponent is given by $\sum_{1 \le i < j \le N} \delta(B_{t_j} - B_{t_i})$ and it does not depend on the mollifier. The easiest mollifier for this computation is

$\tilde{\epsilon}^{-1}\tilde{\varrho}(\tilde{\epsilon}^{-1}t)\epsilon^{-1}\varrho(\epsilon^{-1}x)$ where ϱ is a Gaussian mean zero, variance one. If we take $\tilde{\epsilon} \to 0$ first, we find that $C_\epsilon = \int \epsilon^{-2}\varrho^2(\epsilon^{-1}x)dx = C\epsilon^{-1}$ and the exponent in (112) is $\int_0^t \sum_{i\neq j}(2\epsilon)^{-1}\varrho((2\epsilon)^{-1}(B_i(s)-B_j(s)))ds$. Another way to write it is that $u_\epsilon(t,x_1,\ldots,x_N) = E[z_\epsilon(t,x_1)\cdots z_\epsilon(t,x_N)]$ where z_ϵ satisfies the multiplicative stochastic heat equation with noise smoothed out in space only, with a Gaussian with variance ϵ, satisfies

$$\partial_t u_\epsilon = \left[\tfrac{1}{2}\sum_i \partial_x^2 + \sum_{i<j}\rho_{2\epsilon}(x_i - x_j)\right] u_\epsilon \tag{113}$$

with initial data $u_\epsilon(0,x_1,\ldots,x_k) = \prod_{i=1}^k z_0(x_i)$.

Now if we want a formula for

$$u(t,x_1,\ldots,x_N) := \mathbb{E}\left[\prod_{i=1}^N Z(t,x_i)\right]. \tag{114}$$

where $Z(t,x)$ is the solution of the unmollified stochastic heat equation

$$\partial_t Z = \tfrac{1}{2}\partial_x^2 Z + \xi Z, \qquad Z(0,x) = Z_0(x), \tag{115}$$

we can let $\epsilon \searrow 0$ in the above to get

$$u(t,x_1,\ldots,x_N) = E^{x_1,\ldots,x_N}\left[e^{\int_0^t \sum_{i<j=1}^N \delta(B_i(s)-B_j(s))ds}\prod_{i=1}^k Z_0(B_i(t))\right], \tag{116}$$

where $\int_0^t \delta(x(s))ds$ is the *local time of* $x(\cdot)$ *at* 0, or, equivalently, that $u(t,x_1,\ldots,x_N)$ satisfies the delta-Bose gas

$$\partial_t u = H_N u, \qquad H_N = \tfrac{1}{2}\sum_i \partial_x^2 + \sum_{i<j}\delta(x_i - x_j). \tag{117}$$

Exercise 3.12. Estimate the size of the moments $\mathbb{E}[Z^N(t,x)]$ as follows. 1. Use Ito's formula on the function[9] $\sum_{i<j}|x_i - x_j|$ to obtain

$$\int_0^t \sum_{i<j}\delta(B_i - B_j)ds = \tfrac{1}{2}\sum_{i<j}|B_i(t) - B_j(t)| - \tfrac{1}{2}\sum_{i<j}\int_0^t \operatorname{sgn}(B_i - B_j)(dB_i - dB_j). \tag{118}$$

So

$$E[z^N(t,x)] = E^{x,\ldots,x}\left[e^{-\frac{1}{2}\sum_{i<j}\int_0^t \operatorname{sgn}(B_i-B_j)(dB_i-dB_j)}F(B(t))\right] \tag{119}$$

where $F(B) = e^{\frac{1}{2}\sum_{i<j}|B_i-B_j|}\prod_{i=1}^k z_0(B_i)$. Now if z_0 has support in $[-L,L]$ and is bounded by C then $F \le e^{C'N^2}$ and if z_0 is bounded below by $c>0$ then $F \ge c^N$.

2. Use the antisymmetry to write

$$\sum_{i<j}\int_0^t \operatorname{sgn}(B_i - B_j)(dB_i - dB_j) = \sum_{i=1}^N \int_0^t \sum_{j\neq i}\operatorname{sgn}(B_i - B_j)dB_i \tag{120}$$

[9]Although this function is technically not allowed for Itô's formula, it is very easy to make sense of these computations by mollifying the function $|x|$ near 0 and then removing the mollification. See any proof of Tanaka's formula for details.

3. Use the Itô isometry to obtain

$$E\left[e^{\frac{1}{2}\sum_{i=1}^N\int_0^t\{\sum_{j\neq i}\operatorname{sgn}(B_i-B_j)\}dB_i}\right]=E\left[e^{\frac{1}{8}\sum_{i=1}^N\int_0^t\{\sum_{j\neq i}\operatorname{sgn}(B_i-B_j)\}^2ds}\right]. \tag{121}$$

4. Show that $\sum_{j\neq i}\operatorname{sgn}(B_i-B_j)=N-1-2(k_i-1)$ where $1\leq k_i\leq N$ is the position of B_i when you order $(B_1,\ldots,B_N)$. The sum over i can be taken in any order so

$$\tfrac{1}{8}\sum_{i=1}^N\{\sum_{j\neq i}\operatorname{sgn}(B_i-B_j)\}^2=\tfrac{1}{24}N(N^2-1) \tag{122}$$

and doesn't depend on the B_i's at all.

5. Conclude that

$$\mathbb{E}[Z^k(t,x)]\sim E\left[e^{\frac{1}{8}\sum_{i=1}^N\int_0^t\{\sum_{j\neq i}\operatorname{sgn}(B_i-B_j)\}^2ds}\right]=e^{\frac{1}{24}N(N^2-1)t}. \tag{123}$$

The tilde is because we threw away the term F by upper or lower bounding it by things much smaller than the key term $e^{\frac{t}{24}N^3}$.

In physics books[10] it is said that the KPZ equation cannot be solved in closed form due to its nonlinear character. In 2010, an exact solution was found for the height distribution at a single point, for the narrow wedge initial conditions, which just means $h(t,x)=\log z(t,x)$ where $z_0(x)=\delta_0$. The exact solution [ACQ11; SS10] provides an expression for the generating function,

$$\mathbb{E}\left[\exp\left[-\exp(h(0,t)-\gamma_t s)\right]\right]=\det(1-P_0K_{t,s}P_0)\,, \tag{124}$$

valid for all real $s\in\mathbb{R}$ and $t>0$, where

$$\gamma_t=(t/2)^{1/3}.$$

Because of the affine invariance of white noise, $h(t,x)$ has the same distribution as $h(t,0)$ up to a parabolic shift from the log of the heat kernel. So this gives the one point distributions of KPZ for this special initial condition (multipoint distributions are not known, and one point are known for only a few initial conditions.)

The right hand side of (124) involves a Fredholm determinant over $L^2(\mathbb{R},dx)$. We'll discuss this later in the notes. P_0 projects on $\mathbb{R}_+$, and $K_{t,s}$ is the smoothed Airy kernel

$$K_{t,s}(x,y)=\int_{\mathbb{R}}du\frac{1}{1+\mathrm{e}^{-\gamma_t(u-s)}}\mathrm{Ai}(x+u)\mathrm{Ai}(y+u)\,. \tag{125}$$

Note that the generating function (124) determines uniquely the distribution function $F_t(s)=\mathbb{P}(h(0,t)\leq s)$. There is an explicit inversion formula yielding $F_t(s)$, again involving Fredholm determinants [ACQ11]. Numerically such determinants can be computed very efficiently through approximating the integral kernel of $K_{t,s}$ by a finite-dimensional matrix of the form $K_{t,s}(x_i,x_j)|_{i,j=1,\ldots,n}$ with carefully chosen base points $\{x_j,j=1,...,n\}$ [Bor10]. In the short time limit one returns to the sum (90) to obtain that the initial fluctuations are Gaussian of order $t^{1/4}$. At later times a characteristic peak develops which overshoots somewhat to the left and finally settles, on the scale $t^{1/3}$, at the GUE Tracy-Widom

[10] For example, beginning of Chap. 6 of Barabási A. L. and Stanley H. E., Fractal Concepts in Surface Growth (Cambridge University Press, Cambridge, UK, 1995).

distribution. The formula for the limit distribution is deduced from (124) by substituting $h(x,t)-\gamma_t s$ by $\gamma_t(\gamma_t^{-1}h(x,t)-s)$. Then, taking the limit $t\to\infty$ on both sides, one obtains

$$\lim_{t\to\infty}\mathbb{P}(\gamma_t^{-1}h(0,t)\le s)=\det(1-P_sK_{\mathrm{Ai}}P_s)\,, \tag{126}$$

the GUE Tracy-Widom distribution.

Let's see how the formal derivation (through divergent series!) goes. We want to compute the generating function on the left hand side of (124). We expand in a series and take the expectation through the sum to get

$$\sum_{N=0}^{\infty}\tfrac{(-1)^N}{N!}\mathrm{e}^{-N(s-(t/24))}\mathbb{E}\left[Z(0,t)^N\right]\,. \tag{127}$$

Since the moments grow as in (123) this is definitely illegal. Since we are trying to reconstruct a distribution from moments one needs for uniqueness $E[Z^N]\le e^{CN}$.

Exercise 3.13. Show that $E[X^n]=E[Y^n]$ where $X=e^Z$, $Y=e^{Z_{\text{disc}}}$ with Z a standard normal and $P(Z_{\text{disc}}=k)=Ce^{-k^2/2}$, $k\in\mathbb{Z}$.

For the narrow wedge,

$$\mathbb{E}\left[Z(0,t)^N\right]=(e^{-H_Nt}\delta_{\vec{0}})(\vec{0}). \tag{128}$$

It turns out the eigenvalues/eigenfunctions of the symmetric operator H_N,

$$H_N\Psi_{\underline{\lambda}}=E_{\underline{\lambda}}\Psi_{\underline{\lambda}}, \tag{129}$$

can be computed explicitly. It is what is called *Bethe ansatz solvable*, meaning that the guess

$$\Psi_{\underline{\lambda}}(x_1,...,x_N)=\sum_{p\in\mathcal{P}_N}A_p\prod_{\alpha=1}^{N}\mathrm{e}^{\mathrm{i}\lambda_{p(\alpha)}x_\alpha}\,, \tag{130}$$

where the sum is over all permutations of $1,...,N$ leads to a complete set of eigenfunctions. Of course, one has to solve the *Bethe equations* for the A_p which is not so simple. The wave numbers $\lambda_{p(\alpha)}$ turn out to be quite complicated: They are arranged in M strings, $1\le M\le N$, each one containing n_α elements, $\sum_{\alpha=1}^M n_\alpha=N$. A string has momentum $q_\alpha\in\mathbb{R}$ and n_α purely imaginary rapidities as

$$\lambda^{\alpha,r}=q_\alpha+i\tfrac{1}{2}(n_\alpha+1-2r)\,,\quad r=1,...,n_\alpha\,. \tag{131}$$

The corresponding energy is

$$E_{\underline{\lambda}}=\tfrac{1}{2}\sum_{\alpha=1}^{M}n_\alpha q_\alpha^2-\frac{1}{24}\sum_{\alpha=1}^{M}(n_\alpha^3-n_\alpha)\,. \tag{132}$$

Hence the N-th moment can be written as

$$\mathbb{E}\left[Z(0,t)^N\right]=\sum_{\underline{\lambda}}\Psi_{\underline{\lambda}}^2(\vec{0})\mathrm{e}^{-E_{\underline{\lambda}}t}\,. \tag{133}$$

In mathematics we got somehow used to eigenfunctions being labelled by positive integers. With the appropriate interpretation of the sum over $\underline{\lambda}$,

$$\sum_{\underline{\lambda}}=\sum_{M=1}^{N}\tfrac{1}{M!}\prod_{\alpha=1}^{M}\sum_{n_\alpha=1}^{\infty}\int_{\mathbb{R}}dq_\alpha(2\pi)^{-1}\delta\Big(\sum_{\alpha=1}^{M}n_\alpha,N\Big)\,, \tag{134}$$

To compute the weights $\Psi^2_{\underline{\lambda}}(\vec{0})$ is rather involved [Dot13; CLD14] and only possible for special initial data. One arrives at

$$|\langle 0|\Psi_{\underline{\lambda}}\rangle|^2 = N!\det\left(\frac{1}{\frac{1}{2}(n_\alpha+n_\beta)+\mathrm{i}(q_\alpha-q_\beta)}\right)^M_{\alpha,\beta=1} \tag{135}$$

and the sum over eigenvalues means

Hence the generating function (LHS of (124)) can be written as

$$\sum_{M=0}^{\infty}\frac{(-1)^M}{M!}\prod_{\alpha=1}^{M}\int_{\mathbb{R}}\frac{dq_\alpha}{2\pi}\sum_{n_\alpha=1}^{\infty}(-1)^{n_\alpha-1}\mathrm{e}^{n_\alpha^3 t/24-n_\alpha(\frac{1}{2}q_\alpha^2 t+s)}\det\left(\frac{1}{\frac{1}{2}(n_{\alpha'}+n_{\beta'})+\mathrm{i}(q_{\alpha'}-q_{\beta'})}\right)^M_{\alpha',\beta'=1} \tag{136}$$

with the understanding that the term with $M=0$ equals 1.

Using

$$\frac{1}{\frac{1}{2}(n_\alpha+n_\beta)+\mathrm{i}(q_\alpha-q_\beta)} = \int_0^\infty d\omega_\alpha \mathrm{e}^{-(\frac{1}{2}(n_\alpha+n_\beta)+\mathrm{i}(q_\alpha-q_\beta))\omega_\alpha} \tag{137}$$

and a simple identity for determinants, one arrives at

$$\sum_{M=0}^{\infty}\frac{(-1)^M}{M!}\prod_{\alpha=1}^{M}\int_0^\infty d\omega_\alpha\int_{\mathbb{R}}\frac{dq_\alpha}{2\pi}\sum_{n_\alpha=1}^{\infty}(-1)^{n_\alpha-1}$$
$$\times\mathrm{e}^{n_\alpha^3 t/24-n_\alpha(\mathrm{i}\frac{1}{2}q_\alpha^2 t+s)}\det\left(\mathrm{e}^{-\frac{1}{2}n_{\alpha'}(\omega_{\alpha'}+\omega_{\beta'})-\mathrm{i}q_{\alpha'}(\omega_{\alpha'}-\omega_{\beta'})}\right)^M_{\alpha',\beta'=1}. \tag{138}$$

Exercise 3.14. Show that this can be written as a Fredholm determinant (see (182)). In particular, setting $\gamma_t=(t/2)^{1/3}$, we rescale $s\to\gamma_t s, \omega_\alpha\to\gamma_t\omega_\alpha, q_\alpha\to q_\alpha/\gamma_t$ we obtain the Fredholm determinant of the integral kernel

$$\int_{\mathbb{R}}\frac{dq}{2\pi}\mathrm{e}^{\mathrm{i}q(\omega'-\omega)}\sum_{n=1}^{\infty}(-1)^{n-1}\mathrm{e}^{n^3 t/24-\gamma_t n\left(q^2+\frac{1}{2}(\omega+\omega')-s\right)}. \tag{139}$$

As anticipated, the sum over n is badly divergent.

Note that there is also a nice way to write the formula as a multiple contour integral

$$u(t,x_1,\dots,x_N)=\int_{\alpha_1+i\mathbb{R}}\frac{dz_1}{2\pi i}\cdots\int_{\alpha_n+i\mathbb{R}}\frac{dz_1}{2\pi i}\prod_{1\le A<B\le N}\frac{z_A-z_B}{z_A-z_B-1}\prod_{j=1}^{N}e^{\frac{t}{2}z_j^2+x_jz_j} \tag{140}$$

with $\alpha_1>\alpha_2+1>\cdots>\alpha_N+(N-1)$. But it doesn't really help and summing produces the same difficulty of a divergent sum.

At this point the physicists had to use the input of the rigorously derived formula (from a related, rigorous formula for asymmetric simple exclusion discovered by Tracy and Widom). It turns out the "right" way to sum the divergent series is to invoke the *Airy trick*

$$\mathrm{e}^{\lambda^3 n^3/3} = \int_{-\infty}^{\infty} dy\mathrm{Ai}(y)\mathrm{e}^{\lambda n y}, \tag{141}$$

where the Airy function is defined by

$$\mathrm{Ai}(z)=\frac{1}{2\pi i}\int_{\langle} dw e^{\frac{1}{3}w^3-zw} \tag{142}$$

with $\langle$ the positively oriented contour going from $e^{-i\pi/3}\infty$ to $e^{i\pi/3}\infty$ through 0.

Exercise 3.15. Prove (141) be deforming the Airy contour to $i\mathbb{R}+\lambda n$. Now show that the function $\frac{1}{2\pi i}\int_{\langle} dw\,\cos(2\pi\lambda w)e^{\frac{1}{3}w^3-zw}$ also works.

The illegal move is now to interchange the summation and this integral. Once inside the integral, the summation is easy. Proceeding with this choice, we set $\lambda = t^{1/3}/2 = \gamma_t/2^{2/3}$ and shift y to $y+q^2+\frac{1}{2}(\omega+\omega')$, to obtain the kernel

$$2^{2/3}\int_{-\infty}^{\infty} dq(2\pi)^{-1}\mathrm{e}^{\mathrm{i}q(\omega'-\omega)}\int dy\mathrm{Ai}\big(2^{2/3}(y+q^2+\tfrac{1}{2}(\omega+\omega')\big)\sum_{n=1}^{\infty}(-1)^{n-1}\mathrm{e}^{-\gamma_t n(s-y)} \tag{143}$$

Summing over n and using the identity

$$2^{2/3}\int_{\mathbb{R}} dq(2\pi)^{-1}\mathrm{e}^{2\mathrm{i}qu}\mathrm{Ai}\big(2^{2/3}(q^2+x)\big)=\mathrm{Ai}(x+u)\mathrm{Ai}(x-u)\,, \tag{144}$$

one obtains the result

$$K_{t,s}(\omega,\omega')=\int_{\mathbb{R}} du\mathrm{Ai}(\omega+u)\mathrm{Ai}(\omega'+u)\frac{\mathrm{e}^{\gamma_t(u-s)}}{1+\mathrm{e}^{\gamma_t(u-s)}}\,. \tag{145}$$

We will give a proper proof later in the notes using the discretization of KPZ called qTASEP. There are a couple of other models, for example, ASEP, for which this can be done. The reason it works there is a little mysterious. In these models there are just a few observables for which one can do exact computations, analogues of the moments above. It just happens that for qTASEP and ASEP with the correct microscopic version of narrow wedge initial data, we get lucky and the observables *do* determine the probability distribution.

4. Markov processes

The state space S of our (time homogeneous) Markov process is a locally compact complete separable metric space. For each Borel set $A\subset S$, $x\in S$ and $t\ge 0$ we have a transition probability

$$p_t(x,A)=P(x_t\in A \mid x_0=x) \tag{146}$$

and associated linear operator on bounded continuous functions on S,

$$P_tf(x)=E_x[f(x(t))]=\int f(y)p_t(x,dy) \tag{147}$$

Let $C_0(S)$ be the bounded continuous functions $S\to\mathbb{R}$ which vanish at infinity. It is called a Feller process if $P_t : C_0\to C_0$ and

$$\lim_{t\to 0}P_tf=f \tag{148}$$

in C_0. The topology is uniform convergence on compact sets, which can easily be supplied a metric. The Markov property means that we have the Chapman-Kolmogorov equations

$$\int p(s,x,dy)p(t,y,A)=p(t+s,x,A) \tag{149}$$

which can be written in compact form

$$P_tP_s=P_{t+s}, \tag{150}$$

i.e. it is a *semi-group* with $P_0=I$. *Semi* because it is lacking inverses in general. This suggest one should try to write

$$P_t=e^{tL}. \tag{151}$$

We can define

$$Lf = \lim_{t\to 0} \frac{P_t f - f}{t} \tag{152}$$

at least for $f \in \mathscr{D}$, the domain, consisting of those f for which the limit exists (in C_0 of course). The question is usually can one really identify $\mathscr{D}$ and does it really determine L and the semi-group P_t? Note that it is not very hard to check that if $f \in \mathcal{D}$ then $P_t f \in \mathcal{D}$ and

$$\partial_t P_t f = L P_t f = P_t L f. \tag{153}$$

Note that $u(t,x) = p(t,x,A) = (P_t \mathbf{1}_A)(x)$ and so it should in principle satisfy the *Kolmogorov* or *backward* equation

$$\partial_t u = Lu, \qquad u(0,x) = \mathbf{1}_A(x), \tag{154}$$

though we don't know $\mathbf{1}_A \in \mathcal{D}$ (it often isn't) or at least $P_t \mathbf{1}_A \in \mathcal{D}$, $t > 0$ (it should be.) There is also no general proof of uniqueness for (154). It has to be done on a case by case basis. Of course, if we are talking about some nice class like diffusions on $\mathbb{R}^n$ where the generator is

$$L = \tfrac{1}{2} \sum_{i,j=1}^{d} a_{ij}(x) \partial^2_{x_i x_j} + \sum_{i=1}^{d} b_i(x) \partial_{x_i} \tag{155}$$

where a and b are bounded continuous and $v^T a v \geq \delta |v|^2$, $\delta > 0$ (uniform ellipticity) corresponding to the solution of the SDE

$$dx = adB + bdt \tag{156}$$

then one knows the domain is C^2 and that one has uniqueness for the backward equation (154) (see, e.g. Stroock and Varadhan "Multidimensional diffusion processes").

Now suppose that $P(x_0 \in A) = \mu_0(A)$, i.e. we start our Markov process with distribution μ_0. Then

$$E_\mu[f(x_t)] = \int E_x[(f(x_t)] d\mu_0 = \int P_t f d\mu_0 =: \int f d\mu_t \tag{157}$$

where $\mu_t(A) = P(x_t \in A)$. A natural way to write this is

$$\mu_t = P_t^* \mu_0 \qquad \text{or} \qquad \partial_t \mu_t = L^* \mu_t. \tag{158}$$

If there is a fixed measure ν with $d\mu_t d\nu = f_t$ we could write $\partial_t f_t = L^* f_t$. Note this is an abuse of notation as the operator L^* acts a little differently on the f's than it does on the μ's. This equation for forward evolving densities is called the *Fokker-Planck* or *forward* equation. Obviously it is a lot more restrictive and you can see why probabilists prefer the backward one.

A probability measure ν is called *invariant* (sometimes *stationary*) if

$$P_t^* \nu = 0 \qquad \textit{i.e.} \qquad \int L f d\nu = 0, \qquad f \in \mathcal{D}. \tag{159}$$

It is called *reversible* if

$$\int f L g d\nu = \int g L f d\nu, \qquad f, g \in \mathcal{D}. \tag{160}$$

So reversible is the same thing as the generator being a symmetric operator on $L^2(S, \nu)$. Note this is less than being *self-adjoint* which means that $L^* = L$ *and* the domain of L^* is equal to the domain of L.

There is a nasty problem in semi-group theory that it is usually very troublesome to check things on the whole domain $\mathcal{D}$. Often it is used as a kind of language because the transition probabilities are not very explicit but the generator is easy to write down, at least on a nice subset of the domain. A subset $\mathcal{C}$ of the domain is called a *core* if for every $f \in \mathcal{D}$ we can find $f_n \in \mathcal{C}$ such that $f_n \to f$ and $Lf_n \to Lf$. Generally it is sufficient to prove things like (159) on a core. But it is not always easy to demonstrate that a nice set (like cylinder functions or something) is a core.

A way to prove uniqueness of the backward equation that will be useful to us is the following. The weak version of the backward equation (154) is that for enough nice functions φ,

$$\int_0^t \int \{(-\partial_s + L^*)\varphi\} u d\nu ds = \int \varphi u d\nu \mid_0^t . \tag{161}$$

Suppose we have two weak solutions u_1 and u_2 with $u_1(0,x) = u_2(0,x)$ and we want to show they are equal. Letting $u = u_2 - u_1$ we have

$$\int_0^t \int \{(-\partial_s - L^*)\varphi\} u d\nu ds = -\int \varphi(t,x) u(t,x) d\nu(x). \tag{162}$$

Now suppose that we can solve the equation $(-\partial_s - L^*)\varphi = 0$, $\varphi(t) = 1_A$ for some A, with a φ nice enough that it is admissible in (162). Then we have $\int_A u(t,x) d\nu(x) = 0$ and it is clear that if we can do this for enough A's then we have $u = 0$, i.e. uniqueness.

4.1. **Discrete state space.** When the particle is at $x \in S$ it waits an exponential amount of time of mean $1/c(x,y)$ then jumps to y. It has to be exponential because that is the only waiting time without memory. We say the rate of jumping x to y is $c(x,y)$. It is doing this for all targets y at the same time, so the total rate it jumps out of x is $\sum_y c(x,y)$. This means that starting at x, x_t for small t is y with probability $c(x,y)t + o(t)$ independently for different y and otherwise still at x. So $P_t f(x) = E_x[f(x_t)] = \sum_y c(x,y) f(y) t + (1 - \sum_y c(x,y)) f(x) + o(t)$ and the generator is

$$Lf(x) = \sum_{y \in S} c(x,y)(f(y) - f(x)). \tag{163}$$

Example 4.1 (Poisson process)**.** *S would be $\mathbb{Z}_+$ and $c(x,y) = 1$ if $y = x+1$ and 0 otherwise. The generator is $Lf(n) = f(n+1) - f(n)$ and and $P_t f(n) = \sum_{m=0}^{\infty} \frac{1}{n!} t^n e^{-t} f(n+m)$. To be in the domain we need $t^{-1}(P_t f - f)(n) \to f(n+1) - f(n)$ and clearly C_0 can be the domain.*

It is not hard to check that

$$L^*\mu = \sum_{y \in S} c(y,x)\mu(y) - c(x,y)\mu(x). \tag{164}$$

In particular, if *for each* $y \in S$ the *detailed balance* condition

$$\mu(x)c(x,y) = \mu(y)c(y,x) \tag{165}$$

holds, then μ is invariant (in fact, reversible). Suppose (following Boltzmann) we write

$$\mu(x) = Z_\beta^{-1} e^{-\beta H(x)} \tag{166}$$

where H is the *energy* or *Hamiltonian* of some system. Z_β is the normalization so that it is a probability measure, i.e. $Z_\beta = \sum_{x\in S} e^{-\beta H(x)}$. Then the detailed balance condition can be written

$$e^{-\beta(H(y)-H(x))} = \frac{c(x,y)}{c(y,x)} \tag{167}$$

for which there are many solutions $c(x,y)$, for example, the famous *Metropolis algorithm.* Much hay has been made from the fact that (167) does not involve Z_β. A typical problem: One has an enormous state space, e.g. Huge lattice $\mathbb{L}$ and $S = \mathbb{L}^{\{-1,1\}}$ and are *given* the energy, e.g. Ising $H(\sigma) = \sum_{x\sim y} \sigma_x\sigma_y$, $\sigma \in L^{\{-1,1\}}$ where the sum is over neighbours x and y, and one wants to compute the *free energy* $\log Z_\beta$. This is done by running the Markov process for a long time on the computer.

Exercise 4.2. Suppose $\phi : S \to S$ is one-to-one. Show that if x_t is Markovian then $\phi(x_t)$ is as well. Give a counterexample to show that if ϕ is not one-to-one it can be false. There are special cases where it is true anyway that turn out to be important for integrability.

Exercise 4.3. Show that for $f \in \mathscr{D}$ the following are martingales for the Markov process with generator L:

(1) $f(x_t) - \int_0^t Lf(x_s)ds$
(2) $f(x_t)\exp\{-\int_0^t \frac{Lf}{f}(x_s)ds\}$, $f > 0$
(3) $\exp\{f(x_t) - \int_0^t e^{-f}Le^f(x_s)ds\}$

Exercise 4.4. Let B_t be a standard Brownian motion starting at $z > 0$ at time 0 and let σ be the hitting time of 0. Use reflection to compute $P(\sigma \le s) = 2P(\sqrt{s}Z \ge z)0$ where Z is $N(0,1)$. Now use the Cameron-Martin-Girsanov formula $\frac{dP^b}{dP}_{[0,T]} = e^{\int_0^T bdB - \frac{1}{2}\int_0^T b^2 ds}$ for the Radon-Nikodym derivative of the Brownian motion with drift b with respect the standard Brownian motion to show that if σ is the hitting time of the standard Brownian motion starting at $z > 0$ to the line bt, then

$$P(\sigma \le T) = 1 - \Phi(\frac{z-bT}{\sqrt{T}}) + e^{2zb}\Phi(\frac{-z-bT}{\sqrt{T}}) \tag{168}$$

where $\Phi(x) = \int_{-\infty}^x \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2}dy$.

Exercise 4.5. Let x and y be Markov processes with state spaces S_x and S_y generators L_x and L_y. Show that they are duality with respect to $H : S_x \times S_y \to \mathbb{R}$ if for each y, $H(\cdot, y)$ is in the domain of L_x and for each x, $H(x,\cdot)$ is in the domain of L_y with

$$(L_x H(\cdot,y))(x) = (L_y H(x,\cdot))(y). \tag{169}$$

Example 4.6. *[Continuous time random walk] Let $p(x,y)$ be the rate of jumping from site x to site y in some lattice, or graph; maybe some subset or all of $\mathbb{Z}^d$ which is the state space of the particle; anyway, it is discrete, call it S. The generator is*

$$Lf(x) = \sum_y p(x,y)(f(y) - f(x)). \tag{170}$$

Show that if $p(x,y)$ is doubly stochastic, i.e. $\sum_x p(x,y) = 1$ for all y, then Lebesgue measure is invariant.

Exercise 4.7. Call Δ (discrete Laplace operator) the case of $\mathbb{Z}$ with $p(x,x+1) = p(x,x-1) = 1$ and $p(x,y) = 0$ otherwise. Show that

$$e^{x\Delta}(u_1,u_2) = e^{-2x} I_{|u_2-u_1|}(2x), \tag{171}$$

where $I_n(2x) := \frac{1}{2\pi i}\oint_{\gamma_0} dz\, e^{x(z+z^{-1})}/z^{n+1}$ is the modified Bessel function of the first kind. Here γ_0 is any simple positively oriented contour around $0 \in \mathbb{C}$.

Exercise 4.8 (Independent random walks on lattice)**.** Many independent copies of the random walk from Example 4.6. Let η_x denote the number at site x. Show that the generator is given by

$$Lf(\eta) = \sum_{x,y} p(x,y)\eta_x(f(\eta^{x\to y}) - f(\eta)) \tag{172}$$

where $\eta^{x\to y}$ has one more particle at y and one less at x. Show that in the doubly stochastic case, products of Poisson measures with the same parameter are invariant. Under what conditions are they reversible?

Example 4.9 (Exclusion processes)**.** *The state space is $\mathbb{S} = \{0,1\}^{\mathbb{Z}}$ (or $\mathbb{Z}^d$, or some graph). If $\eta \in \mathbb{S}$, then $\eta_x = 1$ (respectively $\eta_x = 0$) means there is a particle (respectively, no particle) at $x \in \mathbb{Z}$. The particles try to jump like in the previous example, but the jump only happens if there is no particle in the way. The generator is*

$$Lf(\eta) = \sum_{x,y\in\mathbb{Z}} p(x,y)(f(\eta^{x,y}) - f(\eta)) \tag{173}$$

where $\eta^{x,y}$ has occupancy variables at x and y switched. For KPZ stuff we are mostly concerned with $S = \mathbb{Z}$ and finite range jump law. As long as the original walk does not have mean jump size zero it is supposed to be in the KPZ class. The nearest neighbour case means only jumps of size one are allowed. Random walks which only jump by one are called simple. *So $p(x,x+1) = q$ and $p(x,x-1) = 1-q$ if we like. $q = 1/2$ is called symmetric simple exclusion. It is completely solvable. $q \neq 1/2$ is partially solvable. $q = 1$ is the famous totally asymmetric simple exclusion process (TASEP), which is solvable.*

Example 4.10 (q-TASEP)**.** *Let $0 < q < 1$. There are particles $x_1(t) > x_2(t) > \cdots$ on $\mathbb{Z}$. The particles jump to the right one step at rate $1 - q^{\mathrm{gap}}$ where* gap *is the number of empty sites to the next particle (to the right). The generator is*

$$Lf(\vec{x}) = \sum_{i=1}^{\infty} q^{x_{i-1}-x_i+1}(f(\ldots,x_i+1,\ldots) - f(\ldots,x_i,\ldots)). \tag{174}$$

Note that we could have written it with the η notation as in the exclusion example (or written that one with the $\vec{x}$ notation. The state spaces are written a tiny bit differently but isomorphic if there is a last particle to the right. Sometimes one is better and sometimes the other. q-TASEP is a partially solvable model as we will see.

Exercise 4.11. In the previous two examples, show that Bernoulli product measures are invariant. In the case of exclusion with a symmetric p show that the invariance is true for summands in (173) but in the asymmetric case it relies on a telescoping series. The former is called *detailed balance.*

When checking invariant measures in the problems you can assume that cylinder functions are a core for the generator (This is not trivial. For exclusions it is proved in [Lig85].)

Exercise 4.12. Show that q-TASEP is in duality with *q-Bosons* i.e. the process with state space $Y^N = \{\vec{y} \in \mathbb{Z}^N_{\geq 0}\}$ with

$$L^{\mathrm{q-Boson}} f(\vec{y}) = \sum_{i=1}^{N} (1 - q^{y_i})(f(\ldots, y_{i-1}+1, y_i - 1, \ldots) - f(\ldots, y_{i-1}, y_i, \ldots)) \tag{175}$$

through the duality function

$$H(\vec{x}, \vec{y}) = \prod_{i=0}^{N} q^{(x_i+i)y_i}. \tag{176}$$

Exercise 4.13 (Birth-death process). $S = \{0, 1, 2, \ldots\}$ and the process jumps from $n \to n+1$ at rate c and from $n \to n-1$ at rate dn. Write down the generator and show that an appropriate Poisson distribution is invariant. Is it reversible?

Exercise 4.14 (Two type random walks with creation and mutual annihilation). $S = (\mathbb{Z}^2_{\geq 0})^{\mathbb{Z}}$ at every site we have two types of particles η_x and ζ_x. There are two parts of the dynamics. Creation and annihilation in pairs: At rate c, an η and a ζ particle are added to x and at rate $d\eta_x\zeta_x$, one particle of each type is removed from x. Jumping: The η particles jump as continuous time random walks purely to the right, and the ζ particles jump as continous time random walks purely to the left. The generator is $L = L_{\mathrm{cr}} + L_{\mathrm{rw}}$ where

$$\begin{aligned} L_{\mathrm{cr}} f(\eta, \zeta) &= \textstyle\sum_x c(f(\eta^{+,x}, \zeta^{+,x}) - f(\eta, \zeta)) + d\eta_x\zeta_x(f(\eta^{-,x}, \zeta^{-,x}) - f(\eta, \zeta)) \\ L_{\mathrm{rw}} f(\eta, \zeta) &= \textstyle\sum_x r\eta_x(f(\eta^{x\to x+1}, \zeta) - f(\eta, \zeta)) + r\zeta_x(f(\eta, \zeta^{x\to x-1}) - f(\eta, \zeta)) \end{aligned} \tag{177}$$

Show that η_x, ζ_x products of Poissons with parameters ϱ_1, ϱ_2 with $\varrho_1\varrho_2 = c/d$ are invariant for each of L_{cr} and L_{rw}, and therefore for L. Does it even depend on the jump law of the random walks? Show that the adjoint L^* just has $c \leftrightarrow d$ and the directions of the walks of the η and ζ particles reversed.

Exercise 4.15 (Glauber and Kawasaki). $S = \{-1, 1\}^{\mathbb{Z}^d}$. Define

$$\nabla_x f(\eta) = f(\eta^x) - f(\eta) \qquad \text{and} \qquad \nabla_{xy} f(\eta) = f(\eta^{xy}) - f(\eta) \tag{178}$$

where η^x means the occupation variable at x is flipped (from 1 to -1 or from -1 to 1, i.e. its sign is changed) and η^{xy} means the occupation variables at x and y are exchanged. Let μ be a probability measure on S and define symmetric generators through their quadratic forms (*Dirichlet forms*)

$$\langle f, -L^{\mathrm{Glauber}} f\rangle_\mu := \sum_x \int a(\tau_x\eta)(\nabla_x f)^2 d\mu \tag{179}$$

and

$$\langle f, -L^{\mathrm{Kawasaki}} f\rangle_\mu := \sum_{x\sim y} \int a(\tau_x\eta)(\nabla_{xy} f)^2 d\mu \tag{180}$$

where means neighbour. Compute the generators and show that μ is invariant. Compute the special case where μ is the Ising model $Z^{-1}\exp\{-\beta H(\eta)\}$ where $H = -\sum_{x\sim y}\eta_x\eta_y$ and Z is a normalizing constant to make it a probability measure. Does L depend on Z? Why might that be important? Notice that tweaking a gives you a somewhat rich class of dynamics which keeps μ invariant (and reversible), but if you tweak the resulting L in any other way, except for extremely lucky cases (e.g. TASEP) μ is no longer invariant and nobody has the slightest idea how to find the invariant measure, except maybe to prove abstractly that it exists.

5. Fredholm Determinants

If K is an integral operator acting on $H = L^2(X, d\mu)$ through its kernel

$$(Kf)(x) = \int_X K(x,y)f(y)d\mu(y), \tag{181}$$

we define the *Fredholm determinant* by

$$\det(I + \lambda K) = 1 + \sum_{n=1}^{\infty} \frac{\lambda^n}{n!} \int_X \cdots \int_X \det\left[K(x_i, x_j)\right]_{i,j=1}^n d\mu(x_1)\cdots d\mu(x_n). \tag{182}$$

If $|K(x,y)| \le B$ for all x, y, and μ is a finite measure, the *Fredholm series* (182) converges by Hadamard's inequality,

$$|\det(C_1, \ldots, C_n)| \le \|C_1\| \cdots \|C_n\|$$

where $\|C_i\|$ denotes the Euclidean length of the column vector C_i, since the length of the column vector in $[K(x_i, x_j)]_{i,j=1}^n$ is bounded by $Bn^{1/2}$, and hence the n-th summand in (182) is bounded by $\frac{\lambda^n}{n!} B^n n^{n/2}$.

The reason we put a λ in is to keep track of the orders of dependence on the kernel, and to remind ourselves that it is really a series which is supposed to converge for small λ and may possibly be extended.

If one is not familiar with the definition (182) one might even wonder what it has to do with determinants. Take a matrix $K = [K_{ij}]_{i,j=1}^d$, $d < \infty$, and consider the $d \times d$ determinant $\det(I + \lambda K)$. Clearly it is a polynomial of degree d in λ, $\sum_{n=0}^d a_n \lambda^n$, and its coefficients are given by the rule $a_n = \frac{1}{n!}\partial_\lambda^n \det(I + \lambda K)|_{\lambda=0}$. To compute this, use the rule for differentiating determinants,

$$\partial_\lambda \det(C_1, \ldots, C_d) = \sum_{n=1}^{d} \det(C_1, \ldots, \partial_\lambda C_n, \ldots, C_d)$$

and the fact that, in our particular case, $C_n(\lambda) = e_n + \lambda K_{\cdot,n}$ is linear in λ and $C_n(0) = e_n$, the n-th unit vector. The result is

$$\det(I + \lambda K) = 1 + \lambda \sum_{1 \le i \le d} K_{ii} + \lambda^2 \sum_{1 \le i < j \le d} \det \begin{bmatrix} K_{ii} & K_{ij} \\ K_{ji} & K_{jj} \end{bmatrix}$$
$$+ \lambda^3 \sum_{1 \le i < j < k \le d} \det \begin{bmatrix} K_{ii} & K_{ij} & K_{ik} \\ K_{ji} & K_{jj} & K_{jk} \\ K_{ki} & K_{kj} & K_{kk} \end{bmatrix} + \cdots + \lambda^d \det K. \tag{183}$$

Replacing the ordered sums with unordered sums gives a factor $1/n!$, and setting $\lambda = 1$ we can see that this is a special case of (182).

Exercise 5.1. Give an alternate proof of (183) from the basic definition of the determinant

$$\det(I + \lambda K) = \sum_{\sigma \in S_d} \operatorname{sgn}(\sigma) \prod_{i=1}^{d} (I + \lambda K)_{i,\sigma_i} \tag{184}$$

by expanding the product.

Von Koch's idea [Koc92] was that formula (183) for the determinant was the natural one to extend to $d=\infty$. Fredholm replaced the integral operator (181) on $L^2([0,1],dx)$ by its discretization $[\frac{1}{n}K(\frac{i}{n},\frac{j}{n})]_{i,j=1}^n$ to obtain (182), which he then used to characterize the solvability of the integral equation $(I+K)u=f$ via the non-vanishing of the determinant of $I+K$.

One can of course imagine other, more intuitive definitions of the determinant. Perhaps

$$\det(I+K)=\prod_n(1+\lambda_n) \tag{185}$$

where λ_n are the eigenvalues of K, counted with multiplicity. Or

$$\det(I+\lambda K)=e^{\operatorname{tr}\log(I+\lambda A)} \tag{186}$$

with the trace

$$\operatorname{tr}K=\int d\mu(x)\,K(x,x). \tag{187}$$

Of course, these definitions require some smallness condition on K, but at least they make apparent the important fact that the determinant is invariant under conjugation $\det(I+M^{-1}KM)=\det(I+K)$, or, formally equivelently,

$$\det(I+K_1K_2)=\det(I+K_2K_1), \tag{188}$$

(usually referred to as the cyclic property of determinants) as well as the formula

$$\partial_\beta\det(I+K(\beta))=\det(I+K(\beta))\operatorname{tr}((I+K(\beta))^{-1}\partial_\beta K(\beta)) \tag{189}$$

for $K(\beta)$ depending smoothly on a parameter β, which are more or less obvious from (186). We will also need the fact that

$$\det(I+K+R)=\det(I+K)\operatorname{tr}((I+K)^{-1}R),\qquad R\text{ rank one.} \tag{190}$$

To see this, write the left side as $\det(I+K)\det(I+R')$ where $R'=(I+K)^{-1}R$ and use (186). The series expansion $\log(I+A)=\sum_{n=1}^\infty\frac{-1}{n}A^n$ is rather easy when A has rank one.

Exercise 5.2. Is there a similar formula when R has rank two?

We've been awfully cavalier about writing $(I+K)^{-1}$. But Fredholm's whole point is that this is ok as long as $\det(I+K)\neq0$, just as in finite dimensions. The proof is actually just the finite dimensional proof, scaled up. In finite dim. we have *Cramer's rule*, which says that

$$M_{ij}^{-1}=(\det M)^{-1}(-1)^{i+j}\det(\hat{M}^{ij}) \tag{191}$$

where $\hat{M}^{ij}$ is just M with the jth row and ith column removed. In the present setting Fredholm proved that $(1+K)^{-1}=\det(I+K)^{-1}\times$ an analogue of the cofactor matrix above given by

$$\det(I+K)-\lambda\sum_{n=0}^\infty\frac{1}{n!}\int_X\cdots\int_X\det\begin{bmatrix}K(x,y)&K(x,\mathbf{j})\\K(\mathbf{i},y)&K(\mathbf{i},\mathbf{j})\end{bmatrix}d\mu(x_1)\cdots d\mu(x_n). \tag{192}$$

In any case, one has invertibility iff $\det(I+K)\neq0$.

A more modern way to write (182) is

$$\det(I+\lambda K)=\sum_{n=0}^\infty\lambda^n\operatorname{tr}\Lambda^n(K) \tag{193}$$

where $\Lambda^n(K)$ denotes the action of the tensor product $A \otimes \cdots \otimes A$ on the antisymmetric subspace of $H \otimes \cdots \otimes H$. If P_n denotes the projection onto that subspace and $C_n = P_n \Lambda^n(K) P_n$ then

$$C_n(f_1 \otimes \cdots \otimes f_n) = \frac{1}{n!} \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) A f_{\sigma(1)} \otimes \cdots \otimes A f_{\sigma(n)}$$
$$= \frac{1}{n!} \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \int \cdots \int d\mu(y_1) \cdots d\mu(y_n)\, K(x_1, y_{\sigma(1)}) \cdots K(x_n, y_{\sigma(n)}) f_1(y_1) \cdots f_n(y_n)$$

which shows that C_n is an integral operator with kernel $\det\left[K(x_i, x_j)\right]_{i,j=1}^n$ and hence (193) is just a slick way to write (182). The eigenvalues of $\Lambda^n(K)$ are $\lambda_{i_1} \cdots \lambda_{i_n}$, $i_1 < \cdots < i_n$, so $\operatorname{tr} \Lambda^n(K) = \sum_{i_1 < \cdots < i_n} \lambda_{i_1} \cdots \lambda_{i_n}$ and hence $|\operatorname{tr} \Lambda^n(K)| \le \frac{1}{n!} \|K\|_1^n$, which implies

$$|\det(I + \lambda K)| \le e^{\lambda \|K\|_1}. \tag{194}$$

We write $\|A\|_{\mathrm{op}}$ for the operator norm $\sup_{\|f\|=1} \|Af\|$.

Lemma 5.3. *1. A symmetric non-negative definite operator A has unique non-negative definite square root $\sqrt{A}$. 2. A bounded operator has a unique absolute value $|K| = \sqrt{K^* K}$.*

Proof. Since A is non-negative definite it has a self-adjoint extension. Then we can use the spectral theorem to definite its square root. □

The natural notion of smallness for Fredholm determinants turns out to be the trace norm on operators

$$\|K\|_1 := \operatorname{tr} |K|. \tag{195}$$

This makes sense as the condition for convergence of (185) is $\sum_n |\lambda_n| < \infty$.

Exercise 5.4. Show that an operator with finite trace norm is compact.

Using the Parseval relation, one can check that for such operators the trace can be defined as

$$\operatorname{tr} K = \sum_{n=1}^{\infty} \langle e_n, K e_n \rangle, \tag{196}$$

as it is basis independent. This works for operators on any separable Hilbert space, and in our setting it can be shown that this definition of trace coincides with (187) for K of trace class.

The *Hilbert-Schmidt norm* is easier to compute,

$$\|K\|_2 = (\operatorname{tr} K^* K)^{1/2} = \left(\int dx\, dy\, |K(x,y)|^2 \right)^{1/2}.$$

Exercise 5.5. Show

$$\|K\|_{\mathrm{op}} \le \|K\|_2 \le \|K\|_1, \tag{197}$$

as well as

$$\|K_1 K_2\|_1 \le \|K_1\|_2 \|K_2\|_2, \quad \|AK\|_1 \le \|A\|_{\mathrm{op}} \|K\|_1, \quad \text{and} \quad \|AK\|_2 \le \|A\|_{\mathrm{op}} \|K\|_2,$$

The reason the trace norm is so useful is

Lemma 5.6.

(1) *(Lidskii's Theorem) If K is trace class then* $\mathrm{tr}K = \sum_n \lambda_n$, *where* λ_n *are the eigenvalues of* K. *It follows that the three definitions* (182), (185) *and* (186) *are equivalent.*

(2) $A \mapsto \det(I + A)$ *is continuous in trace norm. Explicitly,*

$$|\det(I + K_1) - \det(I + K_2)| \le \|K_1 - K_2\|_1 \exp(\|K_1\|_1 + \|K_2\|_1 + 1). \tag{198}$$

A proof of Lidskii's theorem can be found in Trace Ideals by Simon. (198) can be proven direcly. Let $f(z) = \det(I + \frac{1}{2}(K_1 + K_2) + z(K_1 - K_2))$, so that the left hand side of (198) is $|f(\frac{1}{2}) - f(-\frac{1}{2})| \le \sup_{-1/2\le t\le 1/2} |f'(t)|$. Cauchy's integral formula $f'(z) = \frac{1}{2\pi i}\oint \frac{f(z')}{z'-z}dz'$ shows that $\sup_{-1/2\le t\le 1/2} |f'(t)| \le \frac{1}{R}\sup_{|z|\le R+\frac{1}{2}} |f(z)|$. By (194),

$$\sup_{|z|\le R+\frac{1}{2}} |f(z)| \le \exp(\tfrac{1}{2}\|K_1 + K_2\|_1 + (R + \tfrac{1}{2})\|K_1 - K_2\|_1)$$

and taking $R = \|K_1 - K_2\|_1^{-1}$ gives (198).

1. (Gaussian distribution) A trivial example is $K(x, y) = \frac{e^{-x^2/2t}}{\sqrt{2\pi t}}$. The operator is rank one, so if P_s is the orthogonal projection from $L^2(\mathbb{R}) \to L^2(s, \infty)$ (which is just multiplication by the function $1_{[s,\infty)}$) then by (186) we have

$$\det(I - P_s K P_s)_{L^2(\mathbb{R},dx)} = 1 - \mathrm{tr}P_s K P_s = \int_{-\infty}^{s} dx\, \frac{e^{-x^2/2t}}{\sqrt{2\pi t}}. \tag{199}$$

Of course, the Gaussian here could be replaced by an arbitrary density.

2. (GUE) Consider the Airy kernel $K_{\mathrm{Ai}}(x, y) = \int_0^\infty dt \mathrm{Ai}(x + t)\mathrm{Ai}(y + t)$ and let $\mathrm{Ai}_t(x) = \mathrm{Ai}(x + t)$ and $H = -\partial_x^2 + x$. Then $H\mathrm{Ai}_t = -t\mathrm{Ai}_t$, the Ai_t, $t \in \mathbb{R}$ are generalized eigenfunctions of H, and K_{Ai} is the orthogonal projection onto the negative eigenspace of H (see Remark **??**). Using $\mathrm{Ai}''(x) = x\mathrm{Ai}(x)$, we have $\partial_t \frac{\mathrm{Ai}(x+t)\mathrm{Ai}'(y+t)-\mathrm{Ai}'(x+t)\mathrm{Ai}(y+t)}{y-x} = \mathrm{Ai}(x + t)\mathrm{Ai}(y + t)$, which yields the Christoffel-Darboux formula

$$K_{\mathrm{Ai}}(x, y) = \frac{\mathrm{Ai}'(x + t)\mathrm{Ai}(y + t) - \mathrm{Ai}(x + t)\mathrm{Ai}'(y + t)}{y - x}. \tag{200}$$

The GUE Tracy-Widom distribution is given by

$$F_{\mathrm{GUE}}(s) = \det(I - P_s K_{\mathrm{Ai}} P_s)_{L^2(\mathbb{R},dx)}. \tag{201}$$

We want to show the operator in the Fredholm determinant is trace class. There are several ways to do this, but the following way will be useful later. First of all note that it can alternatively be written

$$F_{\mathrm{GUE}}(s) = \det(I - \tau_s K_{\mathrm{Ai}} \tau_{-s})_{L^2(\mathbb{R}_+)} = \det(I - K_{\mathrm{Ai}})_{L^2([s,\infty))}. \tag{202}$$

Now because it is a norm and $K_{\mathrm{Ai}} = \int_{-\infty}^{-s} R_t dt$ where $R_t(x, y) = \mathrm{Ai}(x-t)\mathrm{Ai}(y-t)$ we have

$$\|K_{\mathrm{Ai}}\|_1 \le \int_{-\infty}^{-s} \|R_t\|_1 dt. \tag{203}$$

But R_t is rank one, so its trace norm is easy to compute (there is only one possible eigenfunction). We get

$$\|K_{\mathrm{Ai}}\|_1 \le \int_0^\infty \int_{-\infty}^{-s} \mathrm{Ai}^2(x - t)dxdt, \tag{204}$$

which is finite by the following well-known estimates for the Airy function

$$|\mathrm{Ai}(x)| \le Ce^{-\frac{2}{3}x^{3/2}} \quad \text{for } x > 0, \qquad |\mathrm{Ai}(x)| \le C \quad \text{for } x \le 0. \tag{205}$$

Exercise 5.7. On the face of it, it is not so obvious why (202) would define a probability distribution function.

(1) Use (204) to show that $\lim_{s\to\infty} \det(I - P_s K_{\mathrm{Ai}} P_s) = 1$.
(2) Show that $P_s K P_s$ is a composition of projections, and therefore its eigenvalues satisfy $1 \ge \lambda_1(s) \ge \lambda_2(s) \ge \cdots \ge 0$.
(3) Use the min-max characterization of eigenvalues

$$\lambda_k(s) = \max_{\dim U = k} \min_{f\in U} \frac{\langle f, P_s K_{\mathrm{Ai}} P_s f\rangle}{\langle f, f\rangle}, \tag{206}$$

to show that $\lambda_i(s)$ is non-decreasing as s decreases, and hence $\det(I - P_s K_{\mathrm{Ai}} P_s)$ is non-increasing with decreasing s.
(4) Show that if f is in the negative eigenspace of H, $\langle P_s f, K_{\mathrm{Ai}} P_s f\rangle \to \langle f, K_{\mathrm{Ai}} f\rangle = \langle f, f\rangle$. Use this to show that $\det(I - P_s K_{\mathrm{Ai}} P_s) \searrow 0$ as $s \searrow -\infty$.

3. (GOE) $F_{\mathrm{GOE}}(s) = \det(I - P_s B_0 P_s)_{L^2(\mathbb{R},dx)}$ where $B_0(x,y) = \mathrm{Ai}(x+y)$. The fact that the kernel is trace class is much less obvious in this case. The key is the following identity

Exercise 5.8 (ScAiry (Scaled Airy) formula)**.** Use the contour integral formula for the Airy function to prove that

$$\int_{-\infty}^{\infty} dx \mathrm{Ai}(a+x)\mathrm{Ai}(b-x) = 2^{-1/3}\mathrm{Ai}(2^{-1/3}(a+b)) \tag{207}$$

Now use the method from the previous example to show that the kernel B_0 is trace class.

One reason we start with the Fredholm expansion (182) is that this is the way the determinant usually arises from combinatorial expressions. Sometimes the kernels are not trace class, but this should not bother us so much as long as some version of the formal expression can be shown to converge. Often, it is genuinely difficult to show that the resulting expressions define a probability distribution, and we only know it because they arose this way.

Exercise 5.9. Compute $E_0[e^{-\frac{\lambda}{2}\int_0^1 B^2(s)ds}]$ for Brownian motion: Approximate the time integral by $\frac{1}{n}\sum_{k=1}^n B^2(k/n)$ and write the joint density of $B(k/n)$ as $\frac{\exp\{-\frac{1}{2}x^T K_n^{-1} x\}}{(2\pi)^{n/2}\sqrt{\det K_n}}$ where $K_n(i/n, j/n) = \min(i/n, j/n)$. Compute and take the limit to obtain

$$E_0\left[e^{-\frac{\lambda}{2}\int_0^1 B^2(s)ds}\right] = \frac{1}{\sqrt{\det(I+\lambda K)}} \tag{208}$$

where K has kernel $K(x,y) = \min(x,y)$. Now use the fact that K is the inverse of the symmetric operator $-\partial_x^2$ acting on C^2 functions on $[0,1]$ with $f(0) = f'(1) = 0$ with eigenfunctions $\sqrt{2}\sin(\pi(n+1/2)x)$ to show that $\det(I+\lambda K) = \cosh\sqrt{\lambda}$.

One place Fredholm determinants come up is in *determinantal point processes.*

Definition 5.10. Let $\mathcal{X}$ be a discrete space and let μ be a measure on X. A *determinantal point process* on $\mathcal{X}$ with correlation kernel $K : \mathcal{X}\times\mathcal{X} \to \mathbb{C}$ is a measure $\mathcal{W}$ on the power set of $\mathcal{X}$, integrating to 1 and such that for any distinct points $x_1, \dots, x_n \in \mathcal{X}$,

$$\sum_{\substack{Y \subset \mathcal{X}: \\ \{x_1,\dots,x_n\} \subset Y}} \mathcal{W}(Y) = \det\bigl[K(x_i, x_j)\bigr]_{1 \le i,j \le n} \prod_{k=1}^{n} \mu(x_k), \tag{209}$$

where the sum runs over finite subsets of $\mathcal{X}$.

The determinants on the right-hand side are called *n-point correlation functions* or *joint intensities* and denoted by

$$\varrho^{(n)}(x_1, \dots, x_n) = \det\bigl[K(x_i, x_j)\bigr]_{1 \le i,j \le n}. \tag{210}$$

Exercise 5.11 (Karlin-McGregor formula)**.** Let X_i, $1 \le i \le n$, be i.i.d. (time-inhomogeneous) Markov chains on $\mathbb{Z}$ with transition probabilities $p_{k,\ell}(s,t)$ satisfying $p_{k,k+1}(t, t+1) + p_{k,k-1}(t, t+1) = 1$ for all k and $t > 0$. Let us fix initial states $X_i(0) = k_i$ for $k_1 < k_2 < \cdots < k_n$ such that each k_i is even. Then the probability that at time t the Markov chains are at the states $\ell_1 < \ell_2 < \cdots < \ell_n$, and that no two of the chains intersect up to time t, equals $\det\bigl[p_{k_i,\ell_j}(0,t)\bigr]_{1 \le i,j \le n}$.

(Hint from Varadhan: for a permutation $\sigma \in S_n$ and $0 \le s \le t$, define the process

$$M_\sigma(s) = \prod_{i=1}^{n} P\bigl(X_i(t) = \ell_{\sigma(i)} \big| X_i(s)\bigr), \tag{211}$$

which is a martingale with respect to the filtration generated by the Markov chains X_i. This implies that the process $M = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) M_\sigma$ is also a martingale. Obtain the Karlin-McGregor formula by applying the optional stopping theorem to M for a suitable stopping time.)

Example 5.12 (Non-intersecting random walks)**.** *Let $X_i(t)$, $1 \le i \le n$, be independent time-homogeneous Markov chains on $\mathbb{Z}$ with one step transition probabilities $p_t(x,y)$ satisfying the identity $p_t(x, x-1) + p_t(x, x+1) = 1$ (i.e. every time each random walk makes one unit step either to the left or to the right). Let furthermore $X_i(t)$ be reversible with respect to a probability measure π on $\mathbb{Z}$, i.e. $\pi(x)p_t(x,y) = \pi(y)p_t(y,x)$ for all $x, y \in \mathbb{Z}$ and $t \in \mathbb{N}$. Then, conditioned on the events that the values of the random walks at times 0 and $2t$ are fixed, i.e. $X_i(0) = X_i(2t) = x_i$ for all $1 \le i \le n$ where each x_i is even, and no two of the random walks intersect on the time interval $[0, 2t]$, the configuration of mid-positions $\{X_i(t) : 1 \le i \le n\}$, with fixed t, is a determinantal point process on $\mathbb{Z}$ with respect to the measure π, i.e.*

$$P\bigl[X_i(t) = z_i, 1 \le i \le n\bigr] = \det\bigl[K(z_i, z_j)\bigr]_{1 \le i,j \le n} \prod_{k=1}^{n} \pi(z_k), \tag{212}$$

where the probability is conditioned by the described event (assuming of course that its probability is non-zero). Here, the correlation kernel K is given by

$$K(u, v) = \sum_{i=1}^{n} \psi_i(u) \phi_i(v), \tag{213}$$

where the functions ψ_i and ϕ_i are defined by

$$\psi_i(u) = \sum_{k=1}^{n} (A^{-\frac{1}{2}})_{i,k} \frac{p_t(x_k, u)}{\pi(u)}, \qquad \phi_i(v) = \sum_{k=1}^{n} (A^{-\frac{1}{2}})_{i,k} \frac{p_t(x_k, v)}{\pi(v)},$$

with the matrix A having the entries $A_{i,k} = \frac{p_{2t}(x_i,x_k)}{\pi(x_k)}$. Invertibility of the matrix A follows from the fact that the probability of the condition is non-zero and Karlin-McGregor formula.

Exercise 5.13. Prove that the mid-positions $\{X_i(t) : 1 \leq i \leq n\}$ of the random walks defined in the previous example form a determinantal process with the correlation kernel (213).

Lemma 5.14. *Let $\mathcal{W}$ be a determinantal point process on a discrete set X with a measure μ and with a correlation kernel K. Then for a Borel set $B \subset \mathcal{X}$ one has*

$$\sum_{X \subset \mathcal{X} \setminus B} \mathcal{W}(X) = \det(I - K)_{\ell^2(B,\mu)}. \tag{214}$$

Proof. This is basically the inclusion-exclusion formula. The left hand side of (214) can be written as

$$\begin{aligned}
\sum_{X \subset \mathcal{X}} \mathcal{W}(X) \prod_{x \in X} ((1 - 1_B(x)) &= \sum_{n \geq 0} \frac{(-1)^n}{n!} \sum_{X \subset \mathcal{X}} \mathcal{W}(X) \sum_{\substack{x_1,\ldots,x_n \in X \\ x_i \neq x_j}} \prod_{k=1}^n 1_B(x_k) \\
&= \sum_{n \geq 0} \frac{(-1)^n}{n!} \sum_{\substack{y_1,\ldots,y_n \in B \\ y_i \neq y_j}} \sum_{X \subset \mathcal{X}} \mathcal{W}(X) \sum_{x_1,\ldots,x_n \in X} \prod_{k=1}^n 1_{x_k = y_k} \\
&= \sum_{n \geq 0} \frac{(-1)^n}{n!} \sum_{\substack{y_1,\ldots,y_n \in B \\ y_i \neq y_j}} \sum_{\substack{X \subset \mathcal{X} \\ \{y_1,\ldots,y_n\} \subset X}} \mathcal{W}(X) \\
&= \sum_{n \geq 0} \frac{(-1)^n}{n!} \sum_{\substack{y_1,\ldots,y_n \in B \\ y_i \neq y_j}} \varrho^{(n)}(y_1, \ldots, y_n) \prod_{k=1}^n \mu(y_k)
\end{aligned}$$

which can be written $\sum_{n \geq 0} \frac{(-1)^n}{n!} \int_{B^n} \det\bigl[K(y_i, y_j)\bigr]_{1 \leq i,j \leq n} d\mu(y_1) \cdots d\mu(y_n)$ since it is determinantal. □

6. Polynuclear growth model (PNG)

The state space is (upper semi-continuous) height functions $h : \mathbb{R} \to \mathbb{Z} \cup \{-\infty\}$. The topology local Hausdorff convergence of hypographs, with $\mathbb{Z} \cup \{-\infty\}$ compactified at $-\infty$. The dynamics has two parts; a continuous, deterministic dynamics $h \mapsto \sup_{|y-x| \leq t} h(y)$ and a discontinuous, stochastic dynamics. In the stochastic part of the dynamics, there is a Poisson point process in space-time with rate 2 (this choice of rate is arbitrary, but convenient in formulas). If (t^*, x^*) is such a point, then $h(t^*, x) = h((t^*)^-, x) + \mathbf{1}(x = x^*)$. In other words, in each interval $[x, x + \mathrm{d}x]$, with probability $2\mathrm{d}x\mathrm{d}t$, a *nucleation* appears in $[t, t + \mathrm{d}t]$ and the height function is increased by 1 at x. Both dynamics are running simultaneously, so the continuous part of the dynamics implies that the nucleations spread in both directions deterministically at speed 1.

Almost equivalently, the model can be thought of as a collection of one-dimensional kinks (down steps) and antikinks (up steps) moving on $\mathbb{R}$. They move ballistically at speed one: kinks to the right, antikinks to the left. They annihilate upon collision, and are also produced, in pairs, a kink at x^+, an antikink at x^-, according to a rate 2 space-time

Poisson process. The height $h(t,0)$ is just its value $h(0,0)$ at time 0, plus the number of kinks and antikinks which have crossed the spatial point 0 up to time t.

We will denote the PNG height function at time t by $h(t,x)$, or sometimes $h(t,x;h_0)$ if we want to indicate the initial data.

The *narrow wedge at* $y \in \mathbb{R}$ is defined by $\mathfrak{d}_y(y) = 0$ and $\mathfrak{d}_y(x) = -\infty$ for $x \neq y$. Note that this is a valid initial data for the PNG height function but not really for the kink/antikink version unless we count having an infinite number of antikinks at 0^- and an infinite number of kinks at 0^+ as a valid state.

If we start PNG with a narrow wedge at y, $\mathfrak{d}_y$, we could call the resulting solution at time t, as a function of x, $\mathscr{A}_t(x,y)$. The PNG dynamics (under the natural coupling where one starts with two or more different initial conditions and runs them using the same Poisson noise) preserve the max operation,

$$h(t,x;\max\{h_1,h_2\}) = \max\{h(t,x;h_1), h(t,x;h_2)\}, \tag{215}$$

and hence one has the variational formula

$$h(t,x;h_0) = \sup_y\{\mathscr{A}_t(x,y) + h_0(y)\}. \tag{216}$$

Exercise 6.1. Start from $\mathfrak{d}_0$ and show that the one-point marginals are equivalent to a Poissonized version of the longest increasing subsequence problem[11]. I.e. Consider the problem of finding Lipschitz-1 paths going from $(0,0)$ to $(t,0)$ which pick up a maximal number of space-time Poisson points along the way. All these paths lie inside the rhombus R with vertices $(0,0)$, $(t/2,t/2)$, $(t/2,-t/2)$ and (t,t), and the maximal number of points which they can pick up is exactly $h(t,0)$. Now rotate the picture by $-45°$, let N denote the number of Poisson points inside the square corresponding to R (so that N is a Poisson$[t]$ random variable), and order these N points according to their x coordinate. The y coordinates of these points define a random permutation σ of $\{1,\dots,N\}$, which is clearly chosen uniformly from S_N, and $h(t,0)$ is nothing but the length of the longest increasing subsequence in σ.

We record a few other key properties of PNG. Typically for KPZ class models, PNG is statistically invariant under spatial shifts

$$T_{-y}h(t,x;T_yh_0) \overset{\text{dist}}{=} h(t,x;h_0), \qquad T_yf(x) = f(x-y), \tag{217}$$

and under reflections

$$Rh(t,x;Rh_0) \overset{\text{dist}}{=} h(t,x;h_0), \qquad Rf(x) = f(-x). \tag{218}$$

A more special property, intimately connected to solvability, is *skew time reversal invariance*,

$$P(h(t,x;g) \leq -f) = P(h(t,x;f) \leq -g). \tag{219}$$

Exercise 6.2. Show that (219) is a consequence of (216).

[11]This was the first exact fluctuation found in the KPZ class, by Baik, Deift in Johansson in 1999. They took a limit of a determinantal formula for the distribution which derived by Gessel about 10 years earlier (it took the decade to realize that it was a formula for this model) and obtained in the limit the same GUE Tracy-Widom distribution which had also been discovered, again about 10 years earlier, for the asymptotic distribution of the top eigenvalue of a matrix from the Gaussian Unitary Ensemble.

Unusually among continuous time random growth models, PNG has a *finite propagation speed*: Given everything up to time t, for $s > 0$, $h(t+s,x)$ only depends on $h(t,y)$, $|y-x| \leq s$, and on the points in the background Poisson process inside the space-time region $\{(u,y) \in \mathbb{R}^2 : t \leq u \leq t+s,\ |y-x| \leq t+s-u\}$.

Note that KPZ growth models like this one do not have an invariant measure because there is a non-trivial upward height shift. So we'll talk about invariant measures for the process modulo the absolute height.

Exercise 6.3. Show that if the model from Exercise 4.14 is put on the lattice $\epsilon\mathbb{Z}$ with rates $r = \epsilon^{-1}$, $c = 2$, $d = 2\epsilon^{-2}$ and densities $\varrho_1 = \epsilon\varrho$, $\varrho_2 = \epsilon\varrho^{-1}$ then it converges to the kink/antikink version of PNG as $\epsilon \to 0$.

From Exercises 4.14 and 6.3 we have

Theorem 6.4. *Two-sided continuous time random walks with generators*

$$A_\rho f(h) = \rho(f(h+1) - f(h)) + \rho^{-1}(f(h-1) - f(h) \tag{220}$$

for $0 < \rho < 1$ *are invariant measures modulo the absolute height for the PNG process. Call these invariant measures* ν_ρ*. The adjoint* L^* *of the generator of the PNG with respect to* ν_ρ *is the generator of the process run backward in time (or upside down).*

Now we take the invariant measure with $\rho = 1$ as a reference measure. The height function is a continuous time symmetric random walk with generator Δ as in Exercise 4.7 which we think of as a height function $g(x)$. For $a \leq b$ and a given height function h consider the *hit kernel*,

$$P^{\mathrm{hit}(h)}_{a,b}(u,v) = P(g \text{ hits the hypograph of } h \text{ on } [a,b] \text{ and } g(b) = v \mid g(a) = u). \tag{221}$$

The *hypograph* of a function h is the set $\{(x,y) \in \mathbb{R} \times \mathbb{Z} \colon y \leq h(x)\}$. The reason we talk about hypographs instead of just saying that g hits h is we want to include the narrow wedge h at some point.

Exercise 6.5. Show that a function h is upper semi-continuous if and only if its hypograph is closed.

We will tend to conflate integral kernels $K(u,v)$ and the integral operator $(Kf)(u) = \int K(u,v)f(v)d\mu(v)$ that they represent. In the case of $P^{\mathrm{hit}(h)}_{a,b}(u,v)$ the measure μ is counting measure on $\mathbb{Z}$.

Notice that the hit kernel is a function of h but actually only uses the restriction of h to $[a,b]$. Sometimes it is helpful to extend both g and h to the whole of $\mathbb{R}$; the g is just extend as a two sided random walk and the h we extend by making it $-\infty$ outside the box. Note that this leaves the hit kernel unchanged.

By a locally finite h we mean one where there are a locally finite number of jumps (up or down).

The next lemma turns out to be the key point behind the solvability of PNG.

Lemma 6.6. *Suppose that* h *is locally finite. As long as neither* $u = h(a)$ *or* $v = h(b)$,

$$LP^{\mathrm{hit}(h)}_{a,b} = 2[P^{\mathrm{hit}(h)}_{a,b}, \nabla]. \tag{222}$$

If $u = h(a)$ *or* $v = h(b)$ *then the left hand side vanishes while the right hand side is* $-P^{\mathrm{hit}(h)}_{a,b}(u+1,v)$ *if* $u = h(a)$ *and* $-P^{\mathrm{hit}(h)}_{a,b}(u,v+1)$ *if* $v = h(b)$.

Here ∇ is the symmetric difference operator

$$\nabla f(u) = \tfrac{1}{2}(f(u+1) - f(u-1)), \tag{223}$$

and $[A, B] = AB - BA$ is the *commutator* of A and B.

The commutator formula (222) is actually an simple consequence of skew time reversibility followed by a slightly annoying but also elementary computation.

Consider a random path g in $[a, b]$ which has the same law as g conditioned on $g(a) = u$ and $g(b) = v$. We choose g to be lower semi-continuous, and extend it as ∞ outside the interval $[a, b]$. Then, recalling that $\Phi(g)$ (defined in (**??**)) is the indicator of the event that a path g ever becomes less than or equal to 0, we have

$$P^{\mathrm{hit}(h)}_{a,b}(u, v) = E_{a,u;b,v}[\Phi(g - h)]e^{(b-a)\Delta}(u, v), \tag{224}$$

where $E_{a,u;b,v}$ denotes the expectation with respect to the law of the random path g, and then by skew time reversibility

$$L_h P^{\mathrm{hit(h)}}_{a,b}(u, v) = E_{a,u;b,v} L_h\Phi(g - h))]e^{(b-a)\Delta}(u, v) = E_{a,u;b,v}[L^*_g\Phi(g - h))]e^{(b-a)\Delta}(u, v), \tag{225}$$

The subscripts h and g indicate which height function the generators are acting on. If the expectation were over the invariant measure, the right hand side would now vanish. It vanishes because the terms which come from the pieces of the generator corresponding to the action at a fixed position telescope. This is the common mechanism to have explicit invariant measures for non-reversible systems (see Exercise 4.11).

So it is not hard to guess that with the pinned measure, one gets boundary terms. They can just be computed explicitly and the result is

Lemma 6.7. $E_{a,u;b,v}[L^*_g\Phi(g - h))]e^{(b-a)\Delta}(u, v) = 2[P^{\mathrm{hit}(h)}_{a,b}, \nabla]$.

Proof. We will do the calculation for the discrete model from Exercise 4.14 after which one can obtain the lemma by taking the limit from Exercise 6.3. Now the notation is that $E_{a,u}$ is expectation of the height function, which jumps down for η particles and up for ζ particles, i.e. $h_x - h_{x-1} = \zeta_x - \eta_x$ with $h_a = u$ with respect to the product measure from Exercise 4.14 on the interval $\{a, a + 1, \dots, b\}$. Note that the exercise deals with the conditional measure that $h(b) = v$, multiplied by the probability that $h(b) = v$ so we can write it as $E_{a,u}[LF\mathbf{1}_{h_b)=v}]$.

The generator $L^* = L^*_{\mathrm{rw}} + L^*_{\mathrm{cr}}$.

Exercise 6.8. $E_{a,u}[L^*_{\mathrm{cr}}F\mathbf{1}_{h_b=v}] = 0$.

From the exercise, we only need to compute $E_{a,u}[L^*_{\mathrm{rw}}F\mathbf{1}_{h_b=v}]$. Explicity it is

$$E_{a,u,b,v}[\sum_x \eta_x(F(\eta^{x\to x-1}, \zeta) - F(\eta, \zeta)) + \zeta_x(F(\eta, \zeta^{x\to x+1}) - F(\eta, \zeta))]. \tag{226}$$

Let's just look at the first terms multiplied by η_x. The ζ_x can just be done in exactly the same way with space flipped. If $x, x - 1$ is outside $\{a, \dots, b\}$ then the term just vanishes. If $x, x - 1$ is completely within $\{a, \dots, b\}$ then

$$E_{a,u,b,v}[\eta_x F(\eta^{x\to x-1}, \zeta)] = E_{a,u,b,v}[(\eta_x + 1)\frac{\mu_{a,u,b,v}(\eta^{x-1\to x})}{\mu_{a,u,b,v}(\eta)}F(\eta, \zeta)] = E_{a,u,b,v}[\eta_{x-1}F]. \tag{227}$$

So the sum in (226) over moves completely within $\{a, \dots, b\}$ produces the term

$$E_{a,u,b,v}[(\eta_a - \eta_b + \zeta_b - \zeta_a)F]. \tag{228}$$

Now let's look at one of the four boundary terms,

$$E_{a,u,b,v}[\zeta_b(F(\eta,\zeta^{b\to b+1})-F(\eta,\zeta))]. \tag{229}$$

The second part cancels the ζ_b term in (228). The first term may as well be written as

$$E_{a,u,b,v}[\zeta_b(F(\eta,\zeta^{-,b})]=E_{a,u,b,v-1}[FR] \qquad R=(\zeta_b+1)\tfrac{\mu_{a,u,b,v}(\eta,\zeta^{+,b})}{\mu_{a,u,b,v-1}(\eta,\zeta)}. \tag{230}$$

Exercise 6.9. Compute the Radon-Nikodym factor R and show that it converges to $\frac{e^{(b-a)\Delta}(u,v-1)}{e^{(b-a)\Delta}(u,v)}$ in the limit from Exercise 6.3.

From the exercise we obtain the Lemma in the limit, since the four terms are all essentially the same calculation. □

Lemma 6.10. *Suppose h is locally finite. As long as neither $u=h(x-t)$ or $v=h(x+t)$,*

$$\partial_t P^{\mathrm{hit(h)}}_{x-t,x+t}(u,v)=\Delta P^{\mathrm{hit(h)}}_{x-t,x+t}+P^{\mathrm{hit(h)}}_{x-t,x+t}\Delta. \tag{231}$$

If $u=h(x-t)$ or $v=h(x+t)$ then the left hand side vanishes while the right hand side is $P^{\mathrm{hit}(h)}_{x-t,x+t}(u+1,v)$ if $u=h(x-t)$ and $P^{\mathrm{hit}(h)}_{x-t,x+t}(u,v+1)$ if $v=h(x+t)$.

Proof. Since it cannot hit immediately and the jumps are locally finite we have $P^{\mathrm{hit(h)}}_{x-t,x+t}(u,v)=P^{\mathrm{hit(h_{[a,b]})}}_{x-t,x+t}(u,v)$ for $a>x-t$ small enough and $b<x+t$ large enough, where $h_{[a,b]}$ is set to be $-\infty$ outside $[a,b]$. This can be rewritten as $e^{(a-(x-t))\Delta}P^{\mathrm{hit(h_{[a,b]})}}_{a,b}e^{(b-(x+t)\Delta}$. □

Define for $a<b$,

$$T^h_{a,b}=e^{a\Delta}P^{\mathrm{hit(h)}}_{a,b}e^{-b\Delta}. \tag{232}$$

In this language, (231) becomes

$$\partial_t T^h_{x-t,x+t}=0. \tag{233}$$

For $x\in\mathbb{R}$, and $t\geq 0$ define, with $\tau_r f(u)=f(u+r)$

$$K(t,x,r,u,v;h)=\tau_r e^{-2t\nabla-x\Delta}T^h_{x-t,x+t}e^{2t\nabla+x\Delta}\tau_{-r} \tag{234}$$

Exercise 6.11. You may worry a little about $e^{-a\Delta}$. It isn't really an issue because if we do both (232) and (234) together we get $e^{-(2\nabla+\Delta)t}P^{\mathrm{hit(h)}}_{x-t,x+t}e^{-(2\nabla+\Delta)t}=e^{-2t\nabla_+}P^{\mathrm{hit(h)}}_{x-t,x+t}e^{2t\nabla_-}$ where $\nabla_+f(u)=f(u+1)-f(u)$ and $\nabla_-f(u)=f(u)-f(u-1)$. By considering a Poisson process, show that $e^{t\nabla_+}$ has kernel convolution with $\frac{t^u}{u!}e^{-t}$, $u\geq 0$ and that this is inverted by convolution with $\frac{(-t)^u}{u!}e^t$.

From this definition and the previous two lemmas, we have

Lemma 6.12.

$$\partial_t K=LK. \tag{235}$$

Lemma 6.13. *For all $y\in[x-t,x+t]$, $K(h+1_y)-K(h)$ is rank one.*

Proof. If $y\notin[x-t,x+t]$ it just vanishes. If $y\in[x-t,x+t]$, $P^{\mathrm{hit(h+1_y)}}_{x-t,x+t}(u,v)-P^{\mathrm{hit(h)}}_{x-t,x+t}(u,v)$ is just the probability for the continuous time random walk g starting at u at time $x-t$ to be at $h(y)+1$ at time y and then go to v at time $x+t$. By the Markov property this is the product $p_{y-(x-t)}(u,h(y)+1)p_{x+t-y}(h(y)+1,v)$, which is evidently rank one. □

Theorem 6.14. *$u(t,h)=\det(I-K(t,x,r,h))_{\ell^2(\mathbb{Z}_{>0})}$ satisfies the Kolmogorov backward equation*

$$\partial_t u=Lu. \tag{236}$$

Proof. L is composed of two pieces, a differential operator corresponding to the deterministic part of the dynamics and a creation operator $2\int dy(f(h+1_y)-f(h))$. By the previous lemma, and the action of differential operators on Fredholm determinants we have

$$(\partial_t - L)\det(I-K) = \det(I-K)\,\mathrm{tr}((I-K)^{-1}(\partial_t - L)K) = 0 \tag{237}$$

from (1) of Lemma 6.12. □

One can extend this to multipoints in the following way. For $x_1 < \cdots < x_n$, the *extended kernel* acting on the n-fold product of $\ell^2(\mathbb{Z}_{>0})$ whose i,jth matrix element is, with the notation $\vec{x} = (x_1,\ldots,x_n)$ and $\vec{r} = (r_1,\ldots,r_n)$

$$K^{\mathrm{ext}}_{ij}(t,\vec{x},\vec{r},h) = \tau_{r_i}(-1_{x_i<x_j}e^{(x_j-x_i)\Delta} + e^{-2t\nabla - x_i\Delta}T^h_{x_i-t,x_j+t}e^{2t\nabla+x_j\Delta})\tau_{-r_j}. \tag{238}$$

Here we sometimes are using $P^{\mathrm{hit}(h)}_{a,b}$ with $a > b$. The convention is that this is always 0. Now let

$$u(t,h) = \det(I - K^{\mathrm{ext}})_{(\ell^2(\mathbb{Z}_{>0}))^n} \tag{239}$$

The same proof as Theorem 6.14 but with a lot of extra complications shows that in this multipoint case we also have $(\partial_t - L)u = 0$.

Theorem 6.15. *Let $u(t,h)$ be given by* (239). *Then*

$$u(0,h) = 1_{h(x_i)\le r_i,\ i=1,\ldots,n}. \tag{240}$$

Proof. Setting $t=0$ in (238) we get an upper triangular matrix with diagonal entries $1 - \tau_{r_i}P^{\mathrm{hit}(h)}_{x_i,x_i}\tau_{-r_i}(u,v) = 1_{u=v>h(x_i)-r_i}$. The determinant is just the product of the determinants of these diagonal entries, each one on $\ell^2(\mathbb{Z}>0)$ giving the indicator that $h(x_i)-r_i<0$, i.e. we get $\prod_{i=1}^n 1_{r_i\ge h(x_i)}$. □

Theorem 6.16. *The finite dimensional distributions of the transition probabilities of PNG are given by*

$$P_h(h(t,x_i)\le r_i, i=1,\ldots,n) = \det(I - K^{\mathrm{ext}}(t,\vec{x},\vec{r},h))_{(\ell^2(\mathbb{Z}_{>0}))^n} \tag{241}$$

Proof. We have shown they satisfy the backward equation. So all we need is uniqueness. We use the argument around (162). All we need is to solve the adjoint equation with initial data $1_{h(x_i)=q_i,i=1,\ldots,m}$. But the adjoint equation is just the equation for PNG upside down and the initial condition can be achieved by taking finite differences of initial data of the form $1_{h(x_i)\ge q_i,i=1,\ldots,m}$, for which we have a Fredholm determinant solution. □

Technically, we still have to prove that the kernels are trace class. In the next section we take a limit to the KPZ fixed point, and we will need to prove the kernels are converging in trace norm, which is a somewhat stronger statement. So we wait till there for the proof.

Exercise 6.17 (Narrow wedge initial data)**.** Since the random walk g can only hit the hypograph of $\mathfrak{d}_0$ at the origin, show that

$$e^{-t\Delta}P^{\mathrm{hit}(\mathfrak{d}_0)}_{x-t,x+t}e^{-t\Delta} = \bar{\chi}_0. \tag{242}$$

Following exercise 4.7, show that

$$e^{2t\nabla+x\Delta}(u_1,u_2) = e^{-2x}\frac{1}{2\pi i}\oint_{\gamma_0}\frac{dz}{z^{u_2-u_1+1}}e^{t(z-z^{-1})+x(z+z^{-1})}, \tag{243}$$

where γ_0 is any simple, positively oriented contour around the origin. When $t > |x|$ this kernel can be expressed in terms of the Bessel function of the first kind, $J_n(x) = \frac{1}{2\pi i}\oint_{\gamma_0} dz\, e^{x(z-z^{-1})/2}/z^{n+1}$:

$$e^{2t\nabla + x\Delta}(u_1,u_2) = e^{-2x}\left(\tfrac{t-x}{t+x}\right)^{(u_2-u_1)/2} J_{u_2-u_1}\big(2\sqrt{t^2-x^2}\big). \tag{244}$$

Use this to show that

$$K(u,v) = \left(\tfrac{t+x}{t-x}\right)^{(u-v)/2} B_s(u,v), \qquad B_s(u,v) = \sum_{\ell\le 0} J_{u-\ell}(2s) J_{v-\ell}(2s)$$

with $s = \sqrt{t^2-x^2}$. B_s is called the *discrete Bessel kernel*. Show that it is an *integrable* kernel:

$$B_s(u,v) = \frac{s}{u-v}\left(J_{u-1}(2s)J_v(2s) - J_u(2s)J_{v-1}(2s)\right).$$

Show that a conjugation reduces it to

$$F_r(s) = \det(I - \tau_r B_s \tau_{-r})_{\ell^2(\mathbb{Z}_{>0})}. \tag{245}$$

Note that from [BO00],

$$\det(I - \tau_r B_s \tau_{-r})_{\ell^2(\mathbb{Z}_{>0})} = e^{-s^2}\det(I_{i-j}(2s))_{i,j=0,\dots,r-1}, \tag{246}$$

where the I_n are modified Bessel functions of the first kind. This latter is Gessel's formula.

7. The KPZ fixed point

The KPZ fixed point is a Markov process whose state space is height functions and is 1:2:3 scaling invariant. It sits in the middle of the class, like Brownian motion sits in the middle of the universality class of stochastic processes without long range correlations which scale to it in the diffusive 1:2 scaling. More precisely, for any model in the class, we should have

$$(S_\epsilon h)(t,x) = \epsilon^{1/2} h(\epsilon^{-3/2}t, \epsilon^{-1}x) - C_\epsilon t \tag{247}$$

converges to the KPZ fixed point (possibly with time rescaled by a constant) as ϵ, which is the ratio of microspace to macrospace, goes to 0. Let's see how this works with (234). First of all, note that the underlying space $\mathbb{Z}_{>0}$ in which the Fredholm determinant is taken has the physical meaning of height. So it should be rescaled to $\epsilon^{1/2}\mathbb{Z}_{>0}$ which is approximating $\mathbb{R}_+$. The operators ∇ and Δ act on functions on this space and are now of order $\epsilon^{1/2}$ and ϵ respectively. We can expand

$$\nabla = \epsilon^{1/2}\partial + \epsilon^{3/2}\tfrac{1}{6}\partial^3 + o(\epsilon^{3/2}), \qquad \Delta = \epsilon\partial^2 + o(\epsilon). \tag{248}$$

The operator ∂ acts on functions on $\mathbb{R}$. Now our formula for PNG rescales as

$$P_{\epsilon^{1/2}h_0(\epsilon^{-1}x)}(\epsilon^{1/2}h(\epsilon^{-3/2}x_i, \epsilon^{-1}t) - C_\epsilon t \le r_i) \tag{249}$$

$$= \det(I - \epsilon^{-1}K(\epsilon^{-3/2}t, \epsilon^{-1}\vec{x}, \epsilon^{-1/2}(\vec{r} + C_\epsilon t), \epsilon^{1/2}h_0(\epsilon^{-1}x))_{\ell^2(\epsilon^{1/2}\mathbb{Z}_{>0})}. \tag{250}$$

The ϵ^{-1} in front of the K is because we want the measure on $\epsilon^{-1/2}\mathbb{Z}_{>0}$ to give mass $\epsilon^{1/2}$ to each site, so that it approximates Lebesgue measure. Let's looks at the three terms in our kernel and their approximations up to terms vanishing in ϵ

$$\underbrace{\tau_{\epsilon^{-1/2}(r_i+C_\epsilon t)} e^{-2\epsilon^{-3/2}t\nabla - \epsilon^{-1}x_i\Delta}}_{\tau_{r_i+C_\epsilon t}e^{-2\epsilon^{-1}t\partial-(t/3)\partial^3-x_i\partial^2}} \underbrace{T^{\epsilon^{1/2}h_0(\epsilon^{-1}\cdot)}_{\epsilon^{-1}x-\epsilon^{-3/2}t,\epsilon^{-1}x+\epsilon^{-3/2}t}}_{\mathrm{T}^{\mathfrak{h}_0}} \underbrace{e^{2\epsilon^{-3/2}t\nabla+\epsilon^{-1}x_j\Delta}\tau_{-\epsilon^{-1/2}(r_j+C_\epsilon t)})}_{e^{2\epsilon^{-1}t\partial+(t/3)\partial^3+x_j\partial^2}\tau_{-r_j-C_\epsilon t}}. \tag{251}$$

In the middle one, we have assumed that the diffusively rescaled initial condition has a limit $\mathfrak{h}_0$, and that the operator converges, to what we call the *Brownian scattering transform* of $\mathfrak{h}_0$. We'll discuss this in a minute. In the first (and similarly in the third) terms, there is a huge shift $\tau_{r_i+C_\epsilon t}$ followed by $e^{-2\epsilon^{-1}t\partial} = e^{-2\epsilon^{-1}t\partial} = \tau_{-2\epsilon^{-1}t}$ since we are on $\mathbb{R}$. So the thing is screaming at us to take $C_\epsilon = 2\epsilon^{-1}$. Now the rest is

$$e^{-\frac{t}{3}\partial^3 - x_i\partial^2 + r_i\partial}\mathrm{T}^{\mathfrak{h}_0}e^{\frac{t}{3}\partial^3 + x_j\partial^2 - r_j\partial}. \tag{252}$$

We need to explain what the terms in (252) are. The presentation is a little informal as $\mathrm{T}^{\mathfrak{h}_0}$ doesn't make sense by itself. First of all, for $t \neq 0$, the operators on the outside can just be written as convolution operators

$$e^{\frac{t}{3}\partial^3 + x\partial^2}(u_1, u_2) = \frac{1}{2\pi i}\int_{\langle} dw\, e^{\frac{t}{3}w^3 + xw^2 + (u_1-u_2)w} \tag{253}$$

$$= t^{-1/3}e^{\frac{2x^3}{3t^2} - \frac{(u_1-u_2)x}{t}}\mathrm{Ai}(-t^{-1/3}(u_1-u_2) + t^{-4/3}x^2). \tag{254}$$

Now for $\mathrm{T}^{\mathfrak{h}_0}$. The following is a little informal but explains what is going on. Suppose that

$$\mathfrak{h}_0(x) = \lim_{\epsilon\searrow 0}\epsilon^{1/2}h_0(\epsilon^{-1}x). \tag{255}$$

By Donsker's invariance principle, we are asymptotically computing the probability for a Brownian motion (of variance 2) to hit $\mathfrak{h}$, but surrounded by backward heat equations!

Exercise 7.1 (Flat initial data)**.** Let $\mathfrak{h}_0 = 0$. Use the reflection principle to show that the hit kernel is the reflection operator, and Exercise 5.8 to compute the finite dimensional distributions of the Airy_1 process.

Now let's see how it can make sense of (252) in general. First, suppose that $\mathfrak{h}$ has compact support, which in this world means that it is equal to $-\infty$ outside a box $[-\ell, \ell]$. Suppose $L >> \ell$ and we want to compute

$$\lim_{L\to\infty} e^{-(t/3)\partial^3 - x_i\partial^2 + r_i\partial}e^{-L\partial^2}P^{\mathrm{hit}(\mathfrak{h})}_{-L,L}e^{-L\partial^2}e^{(t/3)\partial^3 + x_j\partial^2 - r_j\partial}. \tag{256}$$

But the Brownian semigroup is $e^{t\partial^2}$, so since the Brownian particle is free on the intervals $[-L, -\ell)$ and $(\ell, L]$ we can write

$$P^{\mathrm{hit}(\mathfrak{h})}_{-L,L} = e^{(L-\ell)\partial^2}P^{\mathrm{hit}(\mathfrak{h})}_{-\ell,\ell}e^{(L-\ell)\partial^2} \tag{257}$$

and (256) becomes, even without taking the limit, the completely well defined,

$$\mathfrak{K}(t, \vec{x}, \vec{r}, \mathfrak{h}) = e^{-(t/3)\partial^3 + (-x_i-\ell)\partial^2 + r_i\partial}P^{\mathrm{hit}(\mathfrak{h})}_{-\ell,\ell}e^{(t/3)\partial^3 + (x_j-\ell)\partial^2 - r_j\partial}. \tag{258}$$

We claim it actually makes sense even if $\mathfrak{h}$ is not compactly supported. The following useful trick is called *splitting*. We will split at 0 because it makes for simple notations, but in applications one often wants to do it at other points and the same idea works.

Recall that the hitting is being done from above, i.e. hitting $\mathfrak{h}$ means entering its hypograph. We can also write, with the obvious notation.

$$P^{\mathrm{hit}(\mathfrak{h})}_{-\ell,\ell} = e^{2\ell\partial^2} - P^{\mathrm{no\,hit}(\mathfrak{h})}_{-\ell,\ell} \tag{259}$$

and slip the two terms into (258). The first one just gets rid of all the ℓ's and lead to a simple expression using (254). Now by the Markov property

$$P^{\text{no hit}(\mathfrak{h})}_{-\ell,\ell}(u,v) = \int_{\mathfrak{h}(0)}^{\infty} dz P^{\text{no hit}(\mathfrak{h})}_{-\ell,0}(u,z) P^{\text{no hit}(\mathfrak{h})}_{0,\ell}(z,v) \tag{260}$$

$$= \int_{\mathfrak{h}(0)}^{\infty} dz (e^{\ell\partial^2} - P^{\text{hit}(\mathfrak{h})}_{-\ell,0})(u,z)(e^{\ell\partial^2} - P^{\text{hit}(\mathfrak{h})}_{0,\ell})(z,v). \tag{261}$$

One of these sees $\mathfrak{h}^+$ which is just the function $\mathfrak{h}$ on $[0,\infty)$ and the other sees $\mathfrak{h}^-$ which is $\mathfrak{h}(-x)$, $x \in [0,\infty)$. Let $\tau_{\mathfrak{h}^+}$ and $\tau_{\mathfrak{h}^-}$ be the hitting times of $\mathfrak{h}^\pm$ of a Brownian motion starting at z. We can write

$$P^{\text{hit}(\mathfrak{h})}_{0,\ell}(z,v) = \int_0^{\ell} P_z(\tau_{\mathfrak{h}^+} \in dx) e^{(\ell-x)\partial^2}(B(x),v). \tag{262}$$

Here B is the hitting Brownian motion. Note that we run the heat kernel from $B(x)$ and not $\mathfrak{h}(x)$ because it really could be smaller (for example, hitting a narrow wedge). Of course, if $\mathfrak{h}$ is continuous they are the same. Now we can put together all of (259),(261),(262) inserting them into (258), noting that every term in each of them contains $e^{\ell\partial^2}$, to get

$$\mathfrak{K}^{\text{ext}}_{ij}(t,\vec{x},\vec{r},\mathfrak{h}) = -e^{(x_j-x_i)\partial^2+(r_i-r_j)\partial}1_{x_i<x_j} + (\mathfrak{S}_{t,x_i,r_i} - \mathfrak{S}^{\mathfrak{h}^-}_{t,x_i,r_i})^*(\mathfrak{S}_{t,-x_j,r_j} - \mathfrak{S}^{\mathfrak{h}^+}_{t,-x_j,r_j}) \tag{263}$$

where

$$\mathfrak{S}^{\mathfrak{h}}_{t,x,r}(z,v) = E_z[\mathfrak{S}_{t,x-\tau_{\mathfrak{h}},r}(B(\tau),v)], \qquad \mathfrak{S}_{t,x,r} = e^{\frac{t}{3}\partial^3+x\partial^2-r\partial} \tag{264}$$

The *KPZ fixed point* is the height function valued Markov process taking with transition probabilities given by the extension from the cylindrical sub-algebra of

$$P_{\mathfrak{h}_0}(\mathfrak{h}(t,x_1)\le r_1,\dots,\mathfrak{h}(t,x_n)\le r_n) = \det\big(I - \mathfrak{K}^{\text{ext}}(t,\vec{x},\vec{r},\mathfrak{h}_0)\big)_{L^2(\mathbb{R}_+)^n}. \tag{265}$$

Note that the exact formula is only for $\mathfrak{h}_0$ deterministic. If one starts with a measure, the best general formula is the integral of the determinant over that measure, which looks rather unmanageable. In fact, there exist formulas for special random initial data (Brownian motion, Brownian for $x \ge 0$ and $-\infty$ for $x < 0$, either plus constant drift,etc.) but they don't seem to fit into the theory presented here.

Let $\mathfrak{h}(t,x;\mathfrak{h}_0)$ denote the KPZ fixed point with initial data $\mathfrak{h}_0$. Conjecturally, the KPZ fixed point is the unique process with the following properties:

Theorem 7.2. (1) (1:2:3 invariance) $\alpha\mathfrak{h}(\alpha^{-3}t,\alpha^{-2}x;\alpha^{-1}\mathfrak{h}_0(\alpha^2x)) \overset{\text{dist}}{=} \mathfrak{h}(t,x;\mathfrak{h}_0)$,$\alpha > 0$.

(2) (Invariance of Brownian motion) *If $B(x)$ is a two-sided Brownian motion with diffusion coefficient 2, then for each $t > 0$, $\mathfrak{h}(t,x;B) - \mathfrak{h}(t,0;B)$ is also two-sided Brownian motion in x with diffusion coefficient 2.*

(3) (Skew time reversibility) $P\big(\mathfrak{h}(t,x;\mathfrak{g}) \le -\mathfrak{f}(x)\big) = P\big(\mathfrak{h}(t,x;\mathfrak{f}) \le -\mathfrak{g}(x)\big)$.

(4) (Stationarity in space) $\mathfrak{h}(t,x+u;\mathfrak{h}_0(x-u)) \overset{\text{dist}}{=} \mathfrak{h}(t,x;\mathfrak{h}_0)$.

(5) (Reflection invariance) $\mathfrak{h}(t,-x;\mathfrak{h}_0(-x)) \overset{\text{dist}}{=} \mathfrak{h}(t,x;\mathfrak{h}_0)$.

(6) (Affine invariance) $\mathfrak{h}(t,x;\mathfrak{h}_0(x)+a+cx) \overset{\text{dist}}{=} \mathfrak{h}(t,x+\frac{1}{2}ct;\mathfrak{h}_0(x)) + a + cx + \frac{1}{4}c^2t$.

Exercise 7.3. Prove (2)-(5) by passing to the limit from PNG, and (1) and (6) directly from the transition probabilities (265).

The *Airy process* is defined as

$$\mathcal{A}(x) = \mathfrak{h}(1, x; \mathfrak{d}_0) + x^2. \tag{266}$$

Exercise 7.4. As in Example 6.17, we know that the scattering transform of the narrow wedge is just the projector onto $\mathbb{R}_-$. Use the kernel in the KPZ fixed point formula (265), to show that the finite dimensional distributions $P(\mathcal{A}(x_i) \le r_i, i = 1, \dots, n)$ of the Airy process are given by the Fredholm determinant on the n fold product of $L^2(\mathbb{R})$ with entries the *extended Airy kernel*, $K_{ij}(u, v) = \int_0^\infty e^{-t(x_i - x_j)} \mathrm{Ai}(u + t)\mathrm{Ai}(v + t)dt$ if $i \ge j$ and $-\int_{-\infty}^0 e^{-t(x_i - x_j)} \mathrm{Ai}(u + t)\mathrm{Ai}(v + t)dt$ if $i < j$, times the indicator function of $[r_j, \infty)$. Show that the process is stationary in time.

The correlations $E[\mathcal{A}(0)\mathcal{A}(x)]$ decay very slowly, like $\mathcal{O}(|x|^{-2})$. This can be proved from the extended kernel formula but it is quite tricky (see Widom "On asymptotics of the Airy process").

From PNG, the KPZ fixed point also inherits the *preservation of max* property: For any $\mathfrak{f}_1, \mathfrak{f}_2$,

$$\mathfrak{h}(t, x; \mathfrak{f}_1 \vee \mathfrak{f}_2) \overset{\text{dist}}{=} \mathfrak{h}(t, x; \mathfrak{f}_1) \vee \mathfrak{h}(t, x; \mathfrak{f}_2). \tag{267}$$

The *Airy sheet* is defined as

$$\mathcal{A}(x, y) = \mathfrak{h}(1, y; \mathfrak{d}_x) + (x - y)^2. \tag{268}$$

Exercise 7.5. Show that the determinantal formula for the KPZ transition probabilities gives a formula for $\mathbb{P}(\mathcal{A}(x_i, y_j) \le \mathfrak{f}(x_i) + \mathfrak{g}(y_j),\ i, j = 1, 2)$, but $\mathfrak{f}(x_i) + \mathfrak{f}(y_j)$ only span a 3-dimensional linear subspace of $\mathbb{R}^4$. So it does *not* determine the joint distribution of $\mathcal{A}(x_i, y_j)$, $i, j = 1, 2$.

Exercise 7.6. Use stationarity in space to show $\mathfrak{h}(1, y; \mathfrak{d}_x) \overset{\text{dist}}{=} \mathfrak{h}(1, y - x; \mathfrak{d}_0)$ and reflection invariance to show $\mathfrak{h}(1, y - x; \mathfrak{d}_0) \overset{\text{dist}}{=} \mathfrak{h}(1, x - y; \mathfrak{d}_0)$. Conclude that

$$\mathcal{A}(x, y) \overset{\text{dist}}{=} \mathcal{A}(y, x). \tag{269}$$

Exercise 7.7. Show that the Airy sheet is *stationary*: For any fixed x_0, y_0,

$$\mathcal{A}(x + x_0, y + y_0) \overset{\text{dist}}{=} \mathcal{A}(x, y). \tag{270}$$

From the max property, the KPZ fixed point satisfies the Hopf-Lax type variational formula, forced by the Airy sheet: For fixed $t > 0$, the following are equal in distribution as functions of x,

$$\mathfrak{h}(t, x; \mathfrak{h}_0) = \sup_{y \in \mathbb{R}} \left\{ t^{1/3}\mathcal{A}(t^{-2/3}x, t^{-2/3}y) - \tfrac{1}{t}(x - y)^2 + \mathfrak{h}_0(y) \right\}. \tag{271}$$

Exercise 7.8. Prove *Johansson's formula*,

$$F_{\mathrm{GOE}}(2^{-1/3}r) = \sup_y \left\{ \mathcal{A}(y) - y^2 \right\}. \tag{272}$$

Exercise 7.9. Let $\hat{\mathcal{A}}(x, y) = \mathcal{A}(x, y) - (x - y)^2$. Show that if $\hat{\mathcal{A}}^1$ and $\hat{\mathcal{A}}^2$ are independent copies and $t_1 + t_2 = t$ are all positive and fixed, then

$$\sup_z \left\{ t_1^{1/3}\hat{\mathcal{A}}^1(t_1^{-2/3}x, t_1^{-2/3}z) + t_2^{1/3}\hat{\mathcal{A}}^2(t_2^{-2/3}z, t_2^{-2/3}y) \right\} \overset{\text{dist}}{=} t^{1/3}\hat{\mathcal{A}}^1(t^{-2/3}x, t^{-2/3}y). \tag{273}$$

How does this compare to the fact that if X_1 and X_2 are independent standard normal random variables then $X_1 + X_2 \overset{\text{dist}}{=} \sqrt{2}X_1$?

7.1. **Trace class convergence.** To rigorously prove the convergence of the PNG formula to the KPZ fixed point formula, one has to show the kernels are converging in trace class. We will content ourselves here with showing that the one point fixed point kernel itself is trace class. The decomposition (263) is really convenient because the kernel is the sum of four terms. We can consider each separately, and the argument is similar for each one, so we only really have to check one of them. They all look like some variant of $(\mathfrak{S}_{t,x,r})^*\mathfrak{S}^{\mathfrak{h}^+}_{t,-x,r}$ and we can just think of this as an integral of rank one operators and estimate the trace norm by the integral of the trace norms of the rank ones, which is just the L^2 norms of the components, just as in (203). This gives

$$\int_{\substack{s\geq 0\\ b\in\mathbb{R},z>\mathfrak{h}_0(0)}} dz\, P_{B(0)=z}(\tau\in ds, B(\tau)\in db)\sqrt{\int_0^\infty du\,|\mathfrak{S}_{t,x,r}(z-u)|^2\int_0^\infty dv\,|\mathfrak{S}_{t,-x-s,r}(b,v)|^2}. \tag{274}$$

Before we start estimating, I want to show that there is actually a non-trivial constraint on the growth of $\mathfrak{h}$ imposed by the trace norm being finite. Suppose that $\mathfrak{h}\sim\gamma x^2$ for large x and is continuous so that $B(\tau)=h(\tau)$. To keep things simple, and without much loss of generality, let's take $t=1$, $x=r=0$. From (254), we have, for large τ, using $\mathrm{Ai}(x)\sim e^{-\frac{2}{3}x^{3/2}}$ for large x, as long as $\gamma<1$,

$$\mathfrak{S}_{1,-\tau,0}(h,v)\sim e^{-\frac{2}{3}\tau^3+\gamma\tau^3-v\tau-\frac{2}{3}(v+(1-\gamma)\tau^2)^{3/2}} \tag{275}$$

The probability for the Brownian motion with diffusion coefficient 2 to first hit such a curve at large τ, is essentially $e^{-\frac{1}{8}\int_0^\tau|\dot{x}|^2ds}$ where x is the straight line from z to h. Pretending $z=0$, this is $e^{-\frac{1}{8}\gamma^2\tau^3}$. Which means we have a battle of exponents upstairs at scale τ^3 and something definitely goes wrong at $\gamma=1$ (actually before that, but the argument is too rough at this point).

Looking back at the variational formula (271) we have to admit we were warned. The process $\mathcal{A}$ is stationary, and while not bounded, should only have logarithmic type growth. If $\mathfrak{h}_0$ grows like γx^2 there is a finite time at which the formula just explodes. In fact the time is just $t=\gamma^{-1}$. And this is not some artifact of some formula no longer holding, the solution is really exploding to ∞. Of course, it is a nice problem (open) to track the fluctuations during this explosion. Presumably they are of Gumbel type.

With the hindsight provided by the example, we makes some practical assumption on $\mathfrak{h}_0$. A nice one is that there is some $A<\infty$ so that

$$\mathfrak{h}_0\leq A(1+|x|). \tag{276}$$

Note that we only need upper bounds on our initial data and there is no lower bound. The same thing is true about the KPZ equation. Since it is the logarithm of the stochastic heat equation starting from non-negative data, the conditions are really upper bounds. Even for the regular heat equation one tends to assume initial data are bounded above by $e^{A(1+|x|)}$ and there are real problems of ill-posedness once one lets the exponent grow quadratically. It turns out (276) is actually preserved by the dynamics, in the sense that at time t there is a new A' which works. So it is a natural (though not optimal) condition to put on our initial data. Furthermore, if we let σ be the hitting time of $A(1+|x|)$ then $\sigma\leq\tau$. For σ we can compute by the reflection principle, for some $\kappa>0$,

$$P_z(\sigma\leq s)\leq Ce^{-\kappa s^{-1}(A+As-z)^2}. \tag{277}$$

and we have

$$\int_0^\infty du\,|\mathfrak{S}_{1,0,0}(z-u)|^2 = \int_0^\infty du\,|\mathrm{Ai}(u-z)|^2 \tag{278}$$

and

$$\int_0^\infty dv\,|\mathfrak{S}_{1,-s,0}(h,v)|^2 = \int_0^\infty dv\,|e^{-\frac{2}{3}s^3+(h(s)-v)s}\mathrm{Ai}(v-h(s)+s^2)|^2. \tag{279}$$

Now using the bound $h(s) \le A(1+s)$ again in these functions it is not hard to check that (274) is bounded.

Exercise 7.10. Use the Fredholm determinant representation to find the right tails of the F_{GUE} and F_{rmGOE} distributions (see Exercises (7.4) and (7.1) for the formulas.)

7.2. **Local properties.** We want to understand the regularity of the KPZ fixed point, in space and time and what its solutions look like locally. It is not hard to see heuristically that they should be locally Brownian in space, i.e. we zoom in $\epsilon^{-1/2}\mathfrak{h}(1,\epsilon x)$ and hope to see Brownian motion. Starting from the narrow wedge, 1:2:3 invariance implies that $\epsilon^{-1/2}\mathfrak{h}(1,\epsilon x;\mathfrak{d}_0) = \mathfrak{h}(\epsilon^{-3/2},x;\mathfrak{d}_0)$, which we expect, by ergodicity, to look spatially like the invariant measure, Brownian motion (always with diffusion coefficient 2). So the scaling relation can turn local properties into global properties and vice-versa. There are several ways to show that the Airy process is locally Brownian. For example, it is the top line of the Airy line ensemble, which has the Brownian Gibbs property. Furthermore, one expects, and can prove that with this information, the variational formula (271) leads to the local Brownian property of essentially any solution. In terms of regularity in t, without loss of generality, we can look at

$$\mathfrak{h}(s+t,0)-\mathfrak{h}(s,0) = \sup_y \left\{t^{1/3}\mathcal{A}(t^{-2/3}y) - \tfrac{1}{t}y^2 + \mathfrak{h}(s,y) - \mathfrak{h}(s,0)\right\}. \tag{280}$$

For t small, y is small because of the term $-\frac{1}{t}y^2$ and we believe $t^{1/3}\mathcal{A}(t^{-2/3}y) \sim B_1(y)$ and $\mathfrak{h}(s,y)-\mathfrak{h}(s,0) \sim B_2(y)$ for independent Brownian motions B_1 and B_2. Letting $B = B_1 + B_2$ we see that the variational problem is essentially

$$\sup_y \left\{B(y) - \tfrac{1}{t}y^2\right\}. \tag{281}$$

Changing $y \mapsto t^\alpha y$ and using Brownian scaling we must have $t^{\alpha/2} = t^{2\alpha-1}$, i.e. $\alpha = 2/3$, which gives

$$\mathfrak{h}(s+t,0)-\mathfrak{h}(s,0) = \mathcal{O}(t^{1/3}), \qquad t \searrow 0. \tag{282}$$

Note that this is an example of the *KPZ relation* between the *transversal* or *wandering exponent* ξ which governs the size of lateral fluctuations, in the sense that the optimal distance is t^ξ, and the fluctuation exponent, ξ which says that the fluctuations are of size t^ξ. In one dimension, we could use the diffusive scaling of the invariant measure, but in general (i.e. any dimension, directed or undirected model) it is believed we still have, by the analogue argument

$$\chi = 2\xi - 1. \tag{283}$$

Once one has (282) it would not be hard to check that $E[(\mathfrak{h}(s+t,0)-\mathfrak{h}(s,0))^p] \le C_p t^{p/3}$ and use the Kolmogorov regularity theorem to conclude that $\mathfrak{h}$ for fairly general initial data is almost surely Hölder β in time for any $\beta < 1/3$.

If we try to check that the solutions of the KPZ fixed point are locally Brownian in space, we find it is hard to see using the extended kernel on the product of $L^2(\mathbb{R}_+)$. The key is an alternative representation, originally due to Prahofer and Spohn in the case of the Airy process.

Let $K_{t,x}$ be the one-point kernel, i.e.

$$P_{\mathfrak{h}_0}(\mathfrak{h}(t,x) \le r) = \det(I - \tau_r K_{t,x}\tau_{-r})_{L^2(\mathbb{R}_+)}. \tag{284}$$

Note that we can also write the determinant as $\det(I - 1_{\ge r} K_{t,x} 1_{\ge r})_{L^2(\mathbb{R})}$. Note also that spatial shifts of $K_{t,x}$ are accomplished through conjugation by heat kernels,

$$e^{(x-y)\partial^2} K_{t,x} e^{(y-x)\partial^2} = K_{t,y}. \tag{285}$$

Proposition 7.11. (Path integral formula for the KPZ fixed point) For $\mathfrak{h}_0 \in \mathrm{UC}$, $t > 0$, and $x_1 < \cdots < x_m$,

$$\begin{aligned} &P_{\mathfrak{h}_0}(\mathfrak{h}(t,x_1) \le r_1, \dots, \mathfrak{h}(t,x_m) \le r_m) \\ &\quad = \det\Big(I - K_{t,x_1} + 1_{\le r_1} e^{(x_2-x_1)\partial^2} 1_{\le r_2} \cdots e^{(x_m-x_{m-1})\partial^2} 1_{\le r_m} e^{(x_1-x_m)\partial^2} K_{t,x_1}\Big)_{L^2(\mathbb{R})}. \end{aligned} \tag{286}$$

Proof. We will give a sketch the proof, skipping technical details such as showing that all operators are trace class, which requires some conjugations omitted here.

In order to simplify notations we will write $\mathcal{W}_{i,j} = e^{(x_j - x_i)\partial^2}$ and use sans-serif notations for the matrix W with these entries. Similarly $K_i = K_{x_i}$. N is the diagonal matrix of multiplication by $1_{>r_i}$. Then $\mathsf{K} = NK^{\mathrm{ext}}N$ can be written as

$$\mathsf{K} = \mathsf{N}(\mathsf{W}^- K + \mathsf{W}^+(K - \mathsf{I}))\mathsf{N}, \tag{287}$$

where K is the diagonal matrix with entries K_i and W^-, W^+ are lower triangular, respectively strictly upper triangular, and defined by

$$\mathsf{W}^-_{ij} = \mathcal{W}_{i,j}\mathbf{1}_{i\ge j}, \qquad \mathsf{W}^+_{ij} = \mathcal{W}_{i,j}\mathbf{1}_{i<j}. \tag{288}$$

Note that the lower triangular terms are solving the heat equation backwards in time, but they always are being applied to K, and as we have seen, this is ok because K is surrounded by operators $\mathfrak{S}_{t,x}$ which can absorb them. The semi-group property implies that

$$\big[(\mathsf{W}^-)^{-1}\big]_{i,j} = I\mathbf{1}_{j=i} - \mathcal{W}_{i,i-1}\mathbf{1}_{j=i-1}. \tag{289}$$

From (285), $\mathcal{W}_{i,j+1}K_{j+1}\mathcal{W}_{j+1,j} = \mathcal{W}_{i,j}K_j$ so

$$\big[(\mathsf{W}^- + \mathsf{W}^+)K(\mathsf{W}^-)^{-1}\big]_{i,j} = \mathcal{W}_{i,j}K_j - \mathcal{W}_{i,j+1}K_{j+1}\mathcal{W}_{j+1,j}\mathbf{1}_{j<n} = \mathcal{W}_{i,n}K_n\mathbf{1}_{j=n}. \tag{290}$$

Only the last column has non-zero entries. To take advantage of it we write

$$\mathsf{I} - \mathsf{K} = (\mathsf{I} + \mathsf{N}\mathsf{W}^+\mathsf{N})\big[\mathsf{I} - (\mathsf{I} + \mathsf{N}\mathsf{W}^+\mathsf{N})^{-1}\mathsf{N}(\mathsf{W}^- + \mathsf{W}^+)K(\mathsf{W}^-)^{-1}\mathsf{W}^-\mathsf{N}\big].$$

Since $\mathsf{N}\mathsf{W}^+\mathsf{N}$ is strictly upper triangular, $\det(\mathsf{I} + \mathsf{N}\mathsf{W}^+\mathsf{N}) = 1$, which in particular shows that $\mathsf{I} + \mathsf{N}\mathsf{W}^+\mathsf{N}$ is invertible. Thus, using (290), we deduce that $\det\big(\mathsf{I} - \mathsf{K}\big)$ is the same as $\det\big(\mathsf{I} - (\mathsf{I} + \mathsf{N}\mathsf{W}^+\mathsf{N})^{-1}\mathsf{N}\mathsf{W}^{(n)}K\mathsf{W}^-\mathsf{N}\big)$ with $\mathsf{W}^{(n)}_{i,j} = \mathcal{W}_{i,n}\mathbf{1}_{j=n}$. Using the cyclic property of the Fredholm determinant we deduce now that $\det(\mathsf{I} - \mathsf{K}) = \det(\mathsf{I} - \widetilde{\mathsf{K}})$ with

$$\widetilde{\mathsf{K}} = K\mathsf{W}^-\mathsf{N}(\mathsf{I} + \mathsf{N}\mathsf{W}^+\mathsf{N})^{-1}\mathsf{N}\mathsf{W}^{(n)}. \tag{291}$$

Since only the last column of $\mathsf{W}^{(n)}$ is non-zero, the same holds for $\widetilde{\mathsf{K}}$, and thus

$$\det(\mathsf{I} - \mathsf{K}) = \det(\mathsf{I} - \widetilde{\mathsf{K}}_{n,n})_{L^2(\mathbb{R})}. \tag{292}$$

Our goal is thus to compute $\widetilde{\mathsf{K}}_{n,n}$. We have, for $0 \le k \le n-i$,

$$\big[(\mathsf{N}\mathsf{W}^{+}\mathsf{N})^k\mathsf{N}\mathsf{W}^{(n)}\big]_{i,n} = \sum_{i<\ell_1<\cdots<\ell_k\le n} N_i\mathcal{W}_{i,\ell_1}N_{\ell_1}\mathcal{W}_{\ell_1,\ell_2}\cdots N_{\ell_{k-1}}\mathcal{W}_{\ell_{k-1},\ell_k}N_{\ell_k}\mathcal{W}_{\ell_k,n},$$

while for $k > n-i$ the left-hand side above equals 0 (the case $k=0$ is interpreted as $N_i\mathcal{W}_{i,n}$). Here we used repeatedly that $N_i^2 = N_i$. Using this to compute the inverse in (291) gives

$$\widetilde{\mathsf{K}}_{i,n} = \sum_{j=1}^{i}\sum_{k=0}^{n-j}(-1)^k \sum_{j=\ell_0<\ell_1<\cdots<\ell_k\le n} K_i\mathcal{W}_{i,j}N_j\mathcal{W}_{j,\ell_1}N_{\ell_1}\mathcal{W}_{\ell_1,\ell_2}N_{\ell_{k-1}}\mathcal{W}_{\ell_{k-1},\ell_k}N_{\ell_k}\mathcal{W}_{\ell_k,n}. \tag{293}$$

Replacing each N_ℓ by $I-\overline{N}_\ell$ except for the first one and simplifying using the inclusion-exclusion formula gives

$$\widetilde{\mathsf{K}}_{i,n} = K_i\mathcal{W}_{i,i+1}\overline{N}_{i+1}\mathcal{W}_{i+1,i+2}\overline{N}_{i+2}\cdots\mathcal{W}_{n-1,n}\overline{N}_n - K_i\mathcal{W}_{i,1}\overline{N}_1\mathcal{W}_{1,2}\overline{N}_2\cdots\mathcal{W}_{n-1,n}\overline{N}_n, \tag{294}$$

and setting $i=n$ yields $\widetilde{\mathsf{K}}_{n,n} = K_n - K_n\mathcal{W}_{n,1}\overline{N}_1\mathcal{W}_{1,2}\overline{N}_2\cdots\mathcal{W}_{n-1,n}\overline{N}_n$, which in view of (292) is the desired result. □

Let's see how to use the path integral formula to compute the multidimensional distributions as we zoom in on a point. By stationarity in space we may as well take the point to be 0.

Proposition 7.12.

$$\lim_{\epsilon\searrow 0} P_{\mathfrak{h}_0}\big(\mathfrak{h}(t,\epsilon x_1)\le r+\epsilon^{1/2}r_1,\ldots,\mathfrak{h}(t,\epsilon x_n)\le r+\epsilon^{1/2}r_n \mid \mathfrak{h}(t,0)=r\big)$$
$$= P(B(x_1)\le r_1,\ldots,B(x_n)\le r_n) \tag{295}$$

Proof. The left hand side is give by

$$B\lim_{\epsilon\searrow 0}\partial_a\,|_{a=r}\, P_{\mathfrak{h}_0}\big(\mathfrak{h}(t,0)=a,\mathfrak{h}(t,\epsilon x_i)\le r+\epsilon^{1/2}r_i, i=1,\ldots,n\big) \tag{296}$$

where $1/B$ is the probability density at r for $\mathfrak{h}(t,0)$. By the determinantal formula (286) the derivative is given by the determinant times the following trace

$$\mathrm{tr}((I-K)^{-1}(\delta_r e^{\epsilon(x_1-x_0)\partial^2}1_{\le r+\epsilon^{1/2}r_1}e^{\epsilon(x_2-x_1)\partial^2}\cdots 1_{\le r+\epsilon^{1/2}r_n}e^{\epsilon(x_0-x_n)\partial^2}K_{t,0})) \tag{297}$$

with the notation $x_0=0$. The δ_r comes from differentiating $1_{\le a}$ and setting $a=r$. We can use the cyclic property of traces to take the $(1-K)^{-1}$ to the end, and writing $J=e^{(x_0-x_n)\partial^2}K_{t,0}(1-K)^{-1}$ the trace is explicitly

$$\int(e^{\epsilon(x_1-x_0)\partial^2}1_{\le r+\epsilon^{1/2}r_1}e^{\epsilon(x_2-x_1)\partial^2}\cdots 1_{\le r+\epsilon^{1/2}r_n})(r,z)J(z,r)dz. \tag{298}$$

By the invariance of the heat kernel under shifts,

$$(e^{\epsilon(x_1-x_0)\partial^2}1_{\le r+\epsilon^{1/2}r_1}\cdots 1_{\le r+\epsilon^{1/2}r_n})(r,z) = (e^{\epsilon(x_1-x_0)\partial^2}1_{\le\epsilon^{1/2}r_1}\cdots 1_{\le\epsilon^{1/2}r_n})(0,z), \tag{299}$$

and since they are diffusively invariant,

$$(e^{\epsilon(x_1-x_0)\partial^2}1_{\le\epsilon^{1/2}r_1}\cdots 1_{\le\epsilon^{1/2}r_n})(0,z) = (e^{(x_1-x_0)\partial^2}1_{\le r_1}\cdots 1_{\le r_n})(0,\epsilon^{-1/2}z), \tag{300}$$

So that the limit in (296) is

$$C\int(e^{(x_1-x_0)\partial^2}1_{\le r_1}\cdots e^{(x_n-x_{n-1})\partial^2}1_{\le r_n})(0,z)dz, \tag{301}$$

where $C = B\partial_1 J(0,r)$. The integral is exactly the right hand side of (295). So $C = 1$ and the proposition is proved. □

Exercise 7.13 (Little research problem)**.** Thms 1.2 and 1.5 of arXiv:0907.0226 contain a determinantal formula for the multipoint distributions in x at time t of the KPZ fixed point starting from a two sided Brownian motion. It should be possible by following the above argument to show that the marginal distribution of the height differences given by the determinantal formula are just those of the invariant Brownian motion. Note that here it is the marginal we are computing and not the conditional as before. The conditional distribution given the height may be interesting.

8. Integrable systems

We begin the topic of integrable system with a very brief account of the history of KdV. Originally it was thought (especially by George Airy[12]) that surface waves in water could be accounted for by the linear equation obtained in the small amplitude approximation,

$$\partial_t\phi + \partial_r^3\phi = 0. \tag{302}$$

This naive point of view was considerably challenged by the observation of *solitary waves*, single waves travelling down long channels without changing their shape. Kortweg and deVries (1895) proposed the non-linear modification

$$\text{(KdV)} \qquad \partial_t\phi + \tfrac{1}{2}\partial_r\phi^2 + \partial_r^3\phi = 0. \tag{303}$$

Exercise 8.1. Show that all bounded travelling wave solutions $\phi(t,x) = f(x-ct)$ of (302) are sinusoidal, while (303) has the solitary wave solutions $3c\,\mathrm{sech}^2(\frac{\sqrt{c}}{2}(x-ct-x_0)$ for any $c > 0$ and x_0.

The equation was more or less forgotten for 70 years until Kruskal and Zabuski began to study it in relation to the famous Fermi-Pasta-Ulam-Tsingou problem. They had been looking for a nice physics problem to check how well the new MANIAC computer (constructed by Metropolis) performed. Fermi suggested they try to show how the spin chain

$$\partial_t^2 q_r = V'(q_{r+1} - q_r) - V'(q_r - q_{r-1}), \qquad r \in \mathbb{Z} \tag{304}$$

thermalizes, in the non-integrable case $V(y) = \frac{1}{2}y^2 + \delta y^4$ (the integrable case $\delta = 0$ is easily solved by Fourier series). Note that this is a Hamiltonian system with

$$H(\vec{q},\vec{p}) = \sum_r \tfrac{1}{2}p_r^2 + V(q_{r+1} - q_r) \tag{305}$$

so the Gibbs measures $Z^{-1}e^{-\beta H}d\vec{q}d\vec{p}$ should be invariant, as well as the microcanonical ensemble which is the uniform measure on $H(\vec{q},\vec{p}) =$ constant. In the absence of extra conserved quantities, one should expect to 'see' the latter after some time (ergodicity). They measured this by looking at the energy in different Fourier modes. Much to their surprise they found that instead of thermalizing, energy kept reappearing in the first few modes.

[12] Airy was also notorious for being slow to communicate to British astronomers the mathematics computations of Adams which predicted the new planet Neptune, based on perturbations in the orbit of Uranus. Le Verrier had been doing identical computations in Paris, and communicated them more quickly to astronomers in Berlin, who found the planet exactly where it was predicted to be, winning the priority race and achieving one of the greatest historical triumphs of mathematics.

Kruskal and Zabusky attempted to explain this by considering one way waves and a continuum limit, reducing it to a first order equation, KdV. They performed numerical experiments on KdV using the special discretization

$$\partial_t \phi_n = (\phi_{n+1} + \phi_n + \phi_{n-1})(\phi_{n+1} - \phi_{n-1}) - (\phi_{n+2} - 2\phi_{n+1} + 2\phi_{n-1} - \phi_{n-2}). \tag{306}$$

Exercise 8.2. Show that the measure with ϕ_n i.i.d. $N(0, \sigma^2)$ is invariant for (306). Show that for the correct value of σ it is also invariant for the very similar

$$d\phi_n = [(\phi_{n+1} + \phi_n + \phi_{n-1})(\phi_{n+1} - \phi_{n-1}) + (\phi_{n+1} - 2\phi_n + \phi_n)]dt + dB_{n+1} - dB_n, \tag{307}$$

where B_n are independent Brownian motions, a spatial discretization of KPZ referred to as *Sasamoto-Spohn model.* Note that neither (306) nor (307) is expected to be integrable in any sense.

Kruskal and Zabusky not only observed the solitary wave solutions, but that when two solitary waves collided, after a brief non-linear dance with each other, they emerged unscathed, with only a phase shift to remember the collision. They named them *solitons.* It was also observed that an initial blob would break up into a train of solitons. This led people to suspect that there were an enormous number of conserved quantities beside the basic *mass* $\int \phi dr$, *momentum* $\int \phi^2 dr$ and *energy*

$$H = -\int_{\mathbb{R}} (\tfrac{1}{6}\phi^3 - \tfrac{1}{2}(\partial_r \phi)^2) dr. \tag{308}$$

After an exciting period where 10 more were found using clever transformations by Miura, and Gardner produced a 'machine' to compute more, Lax made the following observation. If

$$L = -6\partial_r^2 - \phi \qquad \text{and} \qquad B = -4\partial_r^3 - \phi\partial_r - \tfrac{1}{2}\partial_r \phi \tag{309}$$

then $\phi(t)$ evolving according to KdV is equivalent to $L = L(t)$ satisfying

$$\text{(Lax equation)} \qquad \partial_t L + [L, B] = 0. \tag{310}$$

L is symmetric and B is antisymmetric. Therefore

$$\partial_t u = Bu \tag{311}$$

produces an orthogonal fundamental solution U. The Lax equation says that $U(t)^{-1}L(t)U(t)$ is independent of t, in other words, the family $L(t)$ is unitary equivalent, and the KdV flow of $L(t)$ is is *isospectral.* This gives an infinite number of conserved quantities. Whether it is enough to solve the equation starting from some given initial data is a delicate question about to what extent a function can be represented by various spectral data. For nice enough initial data the answer is yes and is given by inverse scattering on the line and inverse spectral theory on the circle (periodic b.c.)

Note that KdV has Hamiltonian H by which we mean that it can be written

$$\partial_t \phi = \mathcal{J} \frac{\delta H}{\delta \phi} \tag{312}$$

where the latter is the Frechet derivative and $\mathcal{J} = \partial_r$ is the skew form, i.e. the Poisson bracket

$$\{F, G\} = \int_{\mathbb{R}} \frac{\delta F}{\delta \phi} \mathcal{J} \frac{\delta G}{\delta \phi} dr \tag{313}$$

is skew symmetric, bilinear and satisfies Leibniz's rule $\{F_1F_2, G\} = \{F_1, G\}F_2 + F_1\{F_2, G\}$ and the Jacobi identity $\{F_1, \{F_2, F_3\}\} + \{F_3, \{F_1, F_2\}\} + \{F_2, \{F_3, F_1\}\} = 0$.

Exercise 8.3. Prove these and show that the evolution of $F(\phi(t))$ is given by $\partial_t F = \{F, H\}$.

Another way to derive Lax's form for KdV is to consider Ψ which simultaneously solves (311) and

$$L\Psi = \lambda\Psi \tag{314}$$

for a constant $\lambda \in \mathbb{C}$. Then on the one hand

$$\partial_t(L\Psi) = (\partial_t L)\Psi + L(\partial_t\Psi) = (\partial_t L + LB)\Psi \tag{315}$$

while on the other hand

$$\partial_t L\Psi = \partial_t\lambda\Psi = \lambda\partial_t\Psi = \lambda B\Psi = B\lambda\Psi = BL\Psi \tag{316}$$

giving (310).

It can also be rewritten as a first order system. Letting $w_1 = \Psi$ and $w_2 = \partial_r\Psi$ we have

$$\partial_r \mathbf{w} = \mathbf{U}\mathbf{w} \qquad\qquad \mathbf{U} = \begin{bmatrix} 0 & 1 \\ -\frac{1}{6}(\phi+\lambda) & 0 \end{bmatrix} \tag{317}$$

and

$$\partial_t w_1 = \partial_t\Psi = B\Psi = -4\partial_r^3\Psi - \phi\partial_r\Psi - \tfrac{1}{2}\partial_r\phi\Psi. \tag{318}$$

But

$$\partial_r^3\Psi = \partial_r(-\tfrac{1}{6}(\phi+\lambda)\Psi) \tag{319}$$

so after a little algebra

$$\partial_t w_1 = \tfrac{1}{6}(\partial_r\phi)w_1 + \tfrac{1}{3}(2\lambda-\phi)w_2. \tag{320}$$

Furthermore

$$\partial_t w_2 = \partial_r\partial_t w_1 = \tfrac{1}{6}(\partial_r\phi)\partial_r w_1 + \tfrac{1}{6}(\partial_r^2\phi)w_1 + \tfrac{1}{3}(2\lambda-\phi)\partial_r w_2 - \tfrac{1}{3}\partial_r\phi w_2 \tag{321}$$

and using $\partial_r w_2 = -\frac{(}{\phi} + \lambda)w_1$ this simplifies to

$$\partial_t w_2 = (-\tfrac{1}{9}\lambda^2 - \tfrac{1}{18}\lambda\phi + \tfrac{1}{18}\phi^2 + \tfrac{1}{6}\partial_r^2\phi)w_1 - \tfrac{1}{6}(\partial_r\phi)w_w \tag{322}$$

or

$$\partial_t \mathbf{w} = \mathbf{V}\mathbf{w} \qquad\qquad V = \begin{bmatrix} \frac{1}{6}\partial_r\phi & \frac{1}{3}(2\lambda-\phi) \\ -\frac{1}{9}\lambda^2 - \frac{1}{18}\lambda\phi + \frac{1}{18}\phi^2 + \frac{1}{6}\partial_r^2\phi & -\frac{1}{6}\partial_r\phi \end{bmatrix} \tag{323}$$

The compatibility condition for $L\Psi = \lambda\Psi$, $\partial_t\Psi = B\Psi$ becomes that there exists a simultaneous fundamental solution $\mathbf{w}(r,t)$ of $\partial_r\mathbf{w} = \mathbf{U}\mathbf{w}$, $\partial_t\mathbf{w} = \mathbf{V}\mathbf{w}$. Now on the one hand we have

$$\partial_t\partial_r\mathbf{w} = \partial_t(\mathbf{U}\mathbf{w}) = (\partial_t\mathbf{U})\mathbf{w} + \mathbf{U}\partial_t\mathbf{w} = (\partial_t\mathbf{U})\mathbf{w} + \mathbf{U}\mathbf{V}\mathbf{w} \tag{324}$$

and on the other hand we have

$$\partial_r\partial_t\mathbf{w} = \partial_r(\mathbf{V}\mathbf{w}) = (\partial_r\mathbf{V})\mathbf{w} + \mathbf{V}\partial_r\mathbf{w} = (\partial_r\mathbf{V})\mathbf{w} + \mathbf{V}\mathbf{U}\mathbf{w}. \tag{325}$$

Equating mixed partials gives us the *zero-curvature* or *Zakharov-Shabat equations*

$$\partial_t\mathbf{U} - \partial_r\mathbf{V} + [\mathbf{U}, \mathbf{V}] = 0. \tag{326}$$

It is called zero-curvature because it is the same equation as the one from differetial geometry if we want to show that the parallel transport of a vector $\mathbf{w}$ along a curve from (r,t) to (r',t') is independent of the path taken: $\partial_r\mathbf{w} = \mathbf{U})\mathbf{w}$ tells you how to parallel transport in the r direction and $\partial_t\mathbf{w} = \mathbf{V}\mathbf{w}$ tells you how to parallel transport in the t direction.

Example 8.4. *The* Kadomtsev–Petviashvili (KPII) equation *is*

$$\partial_t\phi + \tfrac{1}{2}\partial_r\phi^2 + \tfrac{1}{12}\partial_r^3\phi + \tfrac{1}{4}\partial_r^{-1}\partial_x^2\phi = 0. \tag{327}$$

which is shorthand for

$$\partial_r\left(\partial_t\phi + \tfrac{1}{2}\partial_r\phi^2 + \tfrac{1}{12}\partial_r^3\phi\right) + \tfrac{1}{4}\partial_x^2\phi = 0. \tag{328}$$

If ϕ does not depend on x it reduces to the KdV equation. The Lax pair formulation of (327) *is $\partial_t L = [L,B]$ where*

$$L = \partial_x + \partial_r^2 + 2\phi \qquad B = \tfrac{1}{3}\partial_r^3 + \tfrac{2}{3}\phi\partial_r + \tfrac{1}{2}\partial_r\phi - \tfrac{1}{2}\partial_r^{-1}\partial_x\phi. \tag{329}$$

Example 8.5 (Toda lattice)**.** *This is the system* (304) *with*[13]

$$V(y) = e^y - y - 1. \tag{330}$$

Toda discovered this discrete integrable version of KdV, though he was unable to show that it converged to KdV in the limit of small lattice. To put it in Lax form, one introduces Flaschka variables

$$a_r = \tfrac{1}{2}\partial_t\phi_r, \qquad b_r = \tfrac{1}{2}e^{\frac{1}{2}(q_{r+1}-q_r)} \tag{331}$$

and it becomes

$$\partial_t a_r = 2(b_r^2 - b_{r-1}^2) \qquad \partial_t b_r = b_r(a_{r+1} - a_r). \tag{332}$$

Let's look at the system on $r = 0,\dots,N$ with $b_{-1} = b_{N-1} = 0$. So the first equation of (332) *holds for $r = 1,\dots,N-2$ and the second for $r = 0,\dots,N-2$ and at the boundary $\partial_t a_0 = 2b_0^2$ and $\partial_t a_{N-1} = -2b_{N-2}^2$. Now let L be the tri-diagonal (sometimes called* Jacobi*) matrix*

$$L = \begin{bmatrix} a_0 & b_0 & 0 & 0 & 0 & \cdots & 0 \\ b_0 & a_1 & b_1 & 0 & 0 & \cdots & 0 \\ 0 & b_1 & a_2 & b_2 & 0 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \ddots & & \\ 0 & \cdots & 0 & b_{N-4} & a_{N-3} & b_{N-3} & 0 \\ 0 & \cdots & 0 & 0 & b_{N-3} & a_{N-2} & b_{N-2} \\ 0 & \cdots & 0 & 0 & 0 & b_{N-2} & a_{N-1} \end{bmatrix} \tag{333}$$

and let $B = L_+ - L_-$ where L_+ and L_- are the upper and lower triangular parts of L, so that B has b_i on the upper diagonal and $-b_i$ on the lower diagonal. Then the Toda equations can be rewritten in Lax form as

$$\partial_t L + [L,B] = 0. \tag{334}$$

The zero-curvature version is a little different on the lattice since one of the variables is now discrete. It is a compatibility condition for

$$\Psi_{r+1} = U_r(t,\lambda)\Psi_r \qquad \text{and} \qquad \partial_t\Psi_r = V_r(t,\lambda)\Psi_r \tag{335}$$

where in this case (modulo typos)

$$U_r = \begin{pmatrix} p_r + \lambda & e^{q_r} \\ -e^{-q_r} & 0 \end{pmatrix} \qquad \text{and} \qquad V_r = \begin{pmatrix} 0 & -e^{q_r} \\ e^{-q_r} & \lambda \end{pmatrix} \tag{336}$$

and reads

$$\partial_t U_r + U_r V_r - V_{r+1} U_r = 0. \tag{337}$$

[13]usually people take $V(y) = e^y$ but our choice allows the Gibbs measure to be normalizable.

Example 8.6 (2d Toda Lattice)**.** *This is the non-linear wave equation*

$$\partial_\eta \partial_\zeta q = e^{q_r - q_{r-1}} - e^{q_{r+1} - q_r}. \tag{338}$$

It can be written in zero-curvature form

$$\partial_\zeta U - \partial_\eta V + [U, V] = 0 \tag{339}$$

where

$$U = \delta_{i,j-1} + \partial_\zeta q \delta_{i,j} \qquad V = e^{q_r - q_{r-1}} \delta_{i,j+1}. \tag{340}$$

If we take $F = e^q$*,*

$$\eta = \tfrac{1}{2}(t + x) \qquad \text{and} \qquad \zeta = \tfrac{1}{2}(t - x) \tag{341}$$

we can rewrite the equation as

$$\tfrac{1}{4}(\partial_t^2 - \partial_x^2) \log F_r = \frac{F_{r+1} F_{r-1}}{F_r^2} - 1. \tag{342}$$

The Hamiltonian is given by

$$\int_{\mathbb{R}} \sum_r \left(\tfrac{1}{2} p_r^2 + \tfrac{1}{2} (\partial_x q_r)^2 + e^{q_r - q_{r-1}} - 1 \right) dx \tag{343}$$

9. Solution of the 1d Toda lattice

Suppose $Lu = \lambda u$. Then $u_2 = \frac{\lambda - a_0}{b_0} u_1$, $u_3 = \frac{\lambda - a_1}{b_1} u_2 - \frac{b_0}{b_1} u_1$, etc. So the tridiagonal structure with $b_i > 0$ means that for every eigenvalue there is a unique eigenvector. Order the eigenvalues $\lambda_1 < \lambda_2 < \cdots < \lambda_N$ and let $u^{(k)}$ be the normalized eigenvector corresponding to λ_i. Since L is symmetric,

$$U = (u^{(1)}, \ldots, u^{(N)}) \tag{344}$$

is orthogonal. We know that the λ_i are preserved by the Toda flow. We claim that the matrix of eigenvectors satisfies

$$\partial_t U = BU. \tag{345}$$

To see this, write $LU = \Lambda U$ where Λ is the diagonal matrix of eigenvalues. Differentiating gives $(\partial_t L)U + L\partial_t U = \Lambda \partial_t U$, or, using the Lax equation

$$(BL - LB)U = -(L - \Lambda)\partial_t U \qquad \text{or} \qquad (L - \Lambda)(\partial_t U - BU) = 0. \tag{346}$$

But because of the unique eigenvectors, this means that for each k there is an α_k with

$$(\partial_t - B) u^{(k)} = \alpha_k u^{(k)}. \tag{347}$$

Now take the inner product with $u^{(k)}$. Since B is skew-symmetric, we get

$$\tfrac{1}{2} \partial_t \langle u^{(k)}, u^{(k)} \rangle = \alpha_k \langle u^{(k)}, u^{(k)} \rangle. \tag{348}$$

But $\langle u^{(k)}, u^{(k)} \rangle = 1$, so $\alpha_k = 0$, which proves (345).

Now we solve for $u_1^{(k)}$. We know that from (345),

$$\partial_t u_1^{(k)} = (Bu^{(k)})_1 = b_0 u_2^{(k)} \tag{349}$$

and we can use $(Lu^{(k)})_1 = a_0 u_1^{(k)} + b_0 u_2^{(k)} = \lambda_k u_1^{(k)}$, to reduce this to

$$\partial_t u_1^{(k)} = (\lambda_k - a_0) u_1^{(k)}. \tag{350}$$

We can't use this directly to solve for $u_1^{(k)}$ because a_0 is still an unknown part of the solution of Toda. On the other hand, note that all k use the same a_0, so letting $\partial_t n = a_0$

we get $u_1^{(k)}(t) = n(t)e^{\lambda_k t}u_1^{(k)}(0)$. But since U is orthogonal, its rows are also orthonormal and therefore $1 = \sum_k |u_1^{(k)}(t)|^2$ which tells us that

$$|u_1^{(k)}(t)|^2 = \frac{e^{2\lambda_k t}|u_1^{(k)}(0)|^2}{\sum_j e^{2\lambda_j t}|u_1^{(j)}(0)|^2}. \tag{351}$$

We call $w_k(t) = |u_1^{(k)}(t)|^2$. We have associated to the tridiagonal matrix L its spectral data $\lambda_1 < \ldots < \lambda_N$ and weights $w_1, \ldots, w_N$ which satisfy $w_1 + \cdots + w_N = 1$ and as Toda evolves, the eigenvalues remain unchanged and $w_k(t) = \frac{e^{2\lambda_k t w_k(0)}}{\sum_j e^{2\lambda_j t w_j(0)}}$, i.e. the evolution of the spectral data is 'linear'.

Now we explain how to go back from the spectral data to recover the solution of Toda. From $u_2 = \frac{\lambda - a_0}{b_0}u_1$, $u_3 = \frac{\lambda - a_1}{b_1}u_2 - \frac{b_0}{b_1}u_1$, etc. we see that $u_j^{(k)}$ is a polynomial of degree $j-1$ in λ, evaluated at λ_k. The first $N-1$ rows of $(L-\Lambda)U = 0$ are recurrence relations defining u_j/u_1 as a polynomial $p_{j-1}(\lambda)$ of degree $j-1$ with a positive leading coefficient for $j = 2, \ldots, N$. So the matrix U has the form

$$U = \begin{bmatrix} \sqrt{w_1}p_0(\lambda_1) & \sqrt{w_2}p_0(\lambda_2) & \cdots & \sqrt{w_N}p_0(\lambda_N) \\ \sqrt{w_1}p_1(\lambda_1) & \sqrt{w_2}p_1(\lambda_2) & \cdots & \sqrt{w_N}p_1(\lambda_N) \\ \vdots & \vdots & & \vdots \\ \sqrt{w_1}p_{N-1}(\lambda_1) & \sqrt{w_2}p_{N-1}(\lambda_2) & \cdots & \sqrt{w_N}p_{N-1}(\lambda_N) \end{bmatrix}. \tag{352}$$

The columns are an orthonormal basis but so are the rows, and writing this out we have

$$\langle p_m, p_n \rangle_w = \delta_{mn} \qquad \text{where} \qquad \langle f, g \rangle = \int fg d\mu, \qquad \mu(A) = \sum_k w_k 1_A(\lambda_k). \tag{353}$$

$p_0(\lambda) = 1$ and $p_n(\lambda)$ are the normalized orthogonal polynomials with respect to μ. They are unique if we specify that the leading coefficient is positive. They can be obtained from any family $q_k(\lambda) = c_k\lambda^k + \cdots$, $c_k > 0$ by Gram-Schmidt. It is defined inductively. $q_k^{(0)} = q_k$ and, given $q_k^{(n-1)}$, $q_k^{(n)} = q_k^{(n-1)}$ except for $k = n-1$ which is the normalized version of $q_{n-1}^{(n-1)} - \sum_{j=0}^{n-2}\langle q_j^{(n-1)}, q_{n-1}^{(n-1)}\rangle_w q_j^{(n-1)}$. In general, it is not true that Gram-Schmidt gives a result independent of the initial polynomials, but here because we insist they are of increasing degrees with positive leading coefficients, it is true. So from the spectral data we have recovered U. And $L = U\Lambda U^T$ so we have recovered L, and Toda is solved.

Here is a nice way to summarize the solution. When you apply Gram-Schmidt to a matrix M you end up with an orthogonal matrix Q and an upper triangular matrix R. $M = QR$ is called the *QR factorization*. At time 0 we can perform the QR factorization to obtain the transpose of the matrix U as Q with $Q_{ij} = \sqrt{w_i}p_j(\lambda_i)$. The orthogonal polynomials $p_j(\lambda)$ and the weights w_j evolve in time and we call them $p_j(\lambda, t)$ and $w_j(t)$. At time t we need to start with some polynomials to apply Gram-Schmidt with the new weights $w_j(t)$ and a natural choice is $p_j(\lambda, 0)$. But the new matrix we want to apply Gram-Schmidt to is just $D(t)U(0)^T$ where $D(t)$ is the diagonal matrix with $D_{ii}(t) = \sqrt{\frac{w_i(t)}{w_i(0)}}$. So we get

$$D(t)U(0)^T = U(t)^T R(t) \qquad \text{or} \qquad U(0)D(t)U(0)^T = U(0)U(t)^T R(t). \tag{354}$$

Now using that $D(t) = n(t)e^{t\Lambda}$ and $U(0)e^{t\Lambda}U(0)^T = e^{U(0)t\Lambda U(0)^T} = e^{tL(0)}$ we get

$$e^{tL(0)} = Q(t)R(t) \tag{355}$$

with $Q(t) = U(0)U(t)^T$. So given initial $L(0)$ we perform QR factorization on $e^{tL(0)}$ to get (355). The solution of the Toda lattice is

$$L(t) = Q(t)^T L(0) Q(t). \tag{356}$$

10. PNG and 2D Toda

Theorem 10.1. *Let $K_{\eta,\zeta,r}$ be a family of trace class kernels acting on $\ell^2(\mathbb{Z}_{>0})$ and*

$$F(\eta,\zeta,r) = \det(I-K)_{\ell^2(\mathbb{Z}_{>0})}. \tag{357}$$

Suppose that K satisfies for all $u,v>0$,

(1) $K_{r+1}(u,v) = K_r(u+1,v+1)$,
(2) $\partial_\eta K_r(u,v) = K_{r-1}(u+1,v) - K_r(u+1,v)$,
(3) $\partial_\zeta K_r(u,v) = K_{r-1}(u,v+1) - K_r(u,v+1)$.

Then

$$\partial_\eta\partial_\zeta \log F = \frac{F_{r+1}F_{r-1}}{F_r^2} - 1. \tag{358}$$

Before starting the proof, we need to introduce some notations. Let

$$R_r = (I-K_r)^{-1}, \tag{359}$$

and define, for $f \in \ell^2(\mathbb{Z}_{>0})$,

$$\sigma f(u) = f(u+1), \qquad \sigma^* f(u) = f(u-1)\mathbf{1}_{u>1}. \tag{360}$$

Then (1) in the assumptions of the theorem becomes

$$K_{r+1} = \sigma K_r \sigma^*, \tag{361}$$

while

$$\sigma\sigma^* = I \qquad \text{and} \qquad P := I - \sigma^*\sigma \qquad \text{is rank 1.} \tag{362}$$

On the other hand, (2) and (3) in the assumptions of the theorem can be expressed in terms of σ and σ^* as

$$\partial_\eta K = \sigma(K_{r-1}-K_r), \qquad \text{and} \qquad \partial_\zeta K_r = (K_{r-1}-K_r)\sigma^*. \tag{363}$$

By taking the η derivative of the first or the ζ derivative of the second, we also have

$$\partial_\eta\partial_\zeta K_r = K_{r+1} - 2K_r + K_{r-1}. \tag{364}$$

The proof of Thm. 10.1 is based on two identities which we collect in the next result:

Lemma 10.2. (1) $(I+PR_rK_rP)^{-1} = (I-P) + (I-K_r)(I+\sigma^*R_{r+1}\sigma K_r)P$.
(2) $(I+PR_rK_rP)(I+PR_{r-1}K_{r-1}P)^{-1} = I + PR_r(K_r-K_{r-1})(I+\sigma^*R_r\sigma K_{r-1})P$.

Proof. The first step of the proof is to show the following two simple identities:

$$(I+PR_rK_r)^{-1} = (I-K_r)(I+\sigma^*R_{r+1}\sigma K_r), \tag{365}$$

$$(I+PR_rK_rP)^{-1} = (I-P) + (I+PR_rK_r)^{-1}P. \tag{366}$$

For the first one, use the fact that $R_rK_r = K_rR_r = R_r - I$ to write $I + PR_rK_r = I - P + PR_r = (I-\sigma^*\sigma K_r)R_r$. So the inverse is $(1-K_r)(I-\sigma^*\sigma K_r)^{-1}$. But multiplying out $(I-\sigma^*\sigma K_r)(I+\sigma^*R_{r+1}\sigma K_r)$ gives $I - \sigma^*\sigma K_r + \sigma^*R_{r+1}\sigma K_r - \sigma^*\sigma K_r\sigma^*R_{r+1}\sigma K_r = I$.

To get (366), decompose $I+PR_rK_rP$ as $(I+PR_rK_r) - PR_rK_r(I-P)$ and then multiply by $(I-P)+(I+PR_rK_r)^{-1}P$ on the right to get $I - PR_rK_r(I-P)(I+PR_rK_r)^{-1}P$ after some simplifying using that $I-P$ is a projection. Now check that $(I+PR_rK_r)^{-1}P = P(I+R_rK_rP)^{-1}$ by expanding the inverse on both sides as a series to get (366).

(1) follows immediately from (365) and (366). For (2), write the left hand side as

$$I + P(R_rK_r - R_{r-1}K_{r-1})P(I+PR_{r-1}K_{r-1}P)^{-1}$$

and then use (1) to rewrite this as

$$I + P(R_rK_r - R_{r-1}K_{r-1})P(I-K_{r-1})(I+\sigma^*R_r\sigma K_{r-1})P, \tag{367}$$

where we used again the fact that P is a projection. Now use (361) to write

$$(I-P)(I-K_{r-1})(I+\sigma^*R_r\sigma K_{r-1})P = \sigma^*\sigma(I-K_{r-1})(I+\sigma^*R_r\sigma K_{r-1})P \tag{368}$$

$$= (\sigma^*\sigma(I-K_{r-1}) + \sigma^*\sigma K_{r-1})P = 0, \tag{369}$$

and use this to rewrite (367) further as

$$I + P(R_rK_r - R_{r-1}K_{r-1})(I-K_{r-1})(I+\sigma^*R_r\sigma K_{r-1})P. \tag{370}$$

Since $R_{r-1}K_{r-1} = K_{r-1}R_{r-1}$ and $R_rK_r = R_r - I$ we have

$$(R_rK_r - R_{r-1}K_{r-1})(I-K_{r-1}) = R_rK_r(I-K_{r-1}) - K_{r-1} = R_r(K_r - K_{r-1})$$

and using this in (370) shows that it is equal to the desired right hand side. □

Proof of Theorem 10.1. On the one hand we have

$$\partial_\eta\partial_\zeta \log F = \partial_\eta \operatorname{tr}(R_r(K_r - K_{r-1})\sigma^*) \tag{371}$$

$$= \operatorname{tr}(R_r\sigma(K_{r-1}-K_r)R_r(K_r-K_{r-1})\sigma^*) - \operatorname{tr}(R_r\sigma(K_r - 2K_{r-1} + K_{r-2})\sigma^*) \tag{372}$$

On the other hand, from (361) we have from $\det(I-AB) = \det(I-BA)$,

$$F_{r+1} = \det(I - \sigma K_r\sigma^*) = \det(I-(I-P)K_r) \tag{373}$$

But $I-(I-P)K_r = I-(I-P)R_rK_r(I-K_r) = (R_r - (I-P)R_rK_r)(I-K_r)$ and $R_r - (I-P)R_rK_r = I + PR_rK_r$ so

$$F_{r+1} = \det(I+PR_rK_r)F_r \tag{374}$$

and thus

$$\frac{F_{r+1}F_{r-1}}{F_r^2} = \frac{\det(I+PR_rK_r)}{\det(I+PR_{r-1}K_{r-1})} = \det((I+PR_rK_r)(I+PR_{r-1}K_{r-1})^{-1}). \tag{375}$$

From the lemma and because P is rank one, we have

$$\frac{F_{r+1}F_{r-1}}{F_r^2} - 1 = \operatorname{tr}(PR_r(K_r-K_{r-1})(I+\sigma^*R_r\sigma K_{r-1})) \tag{376}$$

Now we take the first P and write it as $P = I - \sigma^*\sigma$ and consider the two terms separately. The one with the $\sigma^*\sigma$ is equal, using $\operatorname{tr} AB = \operatorname{tr} BA$ to (minus)

$$\operatorname{tr}(\sigma R_r(K_r-K_{r-1})\sigma^*) - \operatorname{tr}(\sigma R_r(K_r-K_{r-1})\sigma^*R_rK_r) = \operatorname{tr}(R_r(K_r-K_{r-1})\sigma^*R_r\sigma) \tag{377}$$

so (376) is equal to

$$\operatorname{tr}(R_r(K_r-K_{r-1})(I-\sigma^*R_r\sigma(I-K_{r-1}))) \tag{378}$$

$$= \operatorname{tr}(R_r(K_r-K_{r-1})(I-\sigma^*R_r\sigma(K_r-K_{r-1}))) - \operatorname{tr}(R_r(K_r-K_{r-1})\sigma^*R_r\sigma(I-K_r)) \tag{379}$$

$$= \operatorname{tr}(R_r(K_r-K_{r-1})(I-\sigma^*R_r\sigma(K_r-K_{r-1}))) - \operatorname{tr}((K_r-K_{r-1})\sigma^*R_r\sigma) \tag{380}$$

$$= -\operatorname{tr}(R_r(K_{r+1}-2K_r+K_{r-1})) - \operatorname{tr}(R_r(K_r-K_{r-1})\sigma^*R_r\sigma(K_r-K_{r-1}))). \tag{381}$$

which is the same as (372) after using $\operatorname{tr} AB = \operatorname{tr} BA$ and (361). □

Theorem 10.3. *The one point distribution of PNG, $F(t,x,r) = P_h(h(t,x) \le r)$ satisfy the 2D Toda equation*

$$\partial_\eta \partial_\zeta \log F = \frac{F_{r+1}F_{r-1}}{F_r^2} - 1. \tag{382}$$

for $r > r_0(t,x) := \sup_{|y-x|\le t} h_0(y)$.

Proof. Fix $t \ge 0$, $x \in \mathbb{R}$. For $a \le b$, let $K_r^{a,b}(u,v) = e^{-2t\nabla - x\Delta} T_{a,b}^{h_0} e^{2t\nabla + x\Delta}(u+r, v+r)$, the kernel in the one-point case except that in the scattering transform we remove the dependence on $x-t$ and $x+t$ and replace it by arbitrary parameters a,b. If we let $F_r(t,x) = F_{r,x-t,x+t}(t,x)$ and $F_{r,a,b}(t,x) = \det(I - K_r^{a,b})_{\ell^2(\mathbb{Z}_{>0})}$ for $r \ge r_0(t,x)$, then we find from the previous proof that $\frac{1}{4}(\partial_t^2 - \partial_x^2)\log F_r - \frac{F_{r+1}F_{r-1}}{F_r^2} + 1$ is given by

$$-\big(\partial_{x+t}\partial_a \log F_{r,a,b} - \partial_{x-t}\partial_b \log F_{r,a,b} - \partial_a\partial_b \log F_{r,a,b}\big)\big|_{a=x-t,b=x+t}. \tag{383}$$

To prove that the 2D Toda equation (342) holds, we need to show that this vanishes for $r > r_0(t,x)$. We will show that the left derivative $\partial_{b^-} \log(F_{r,a,b})|_{b=x+t}$ vanishes, the others are the same. Introduce truncated initial data $h_0^{[a,b]}(x) = h_0(x)\mathbf{1}_{x\in[a,b]} - \infty \cdot \mathbf{1}_{x\notin[a,b]}$. If $x-t \le a \le b \le x+t$ then $P_{x-t,x+t}^{\mathrm{hit}(h_0^{[a,b]})} = e^{(a-x+t)\Delta} P_{a,b}^{\mathrm{hit}(h_0)} e^{(x+t-b)\Delta}$ because it cannot hit $\mathrm{hypo}(\mathrm{h}_0^{[a,b]})$ outside $[a,b]$. In other words, when $[a,b] \subseteq [x-t,x+t]$, the scattering transform for h_0 in the interval $[a,b]$ is the same as for the truncated initial data $h_0^{[a,b]}$ in the interval $[x-t,x+t]$, and thus

$$F_{r,a,b}(t,x) = F_r(t,x;h_0^{[a,b]}) = \mathbb{P}_{h_0^{[a,b]}}\big(h(t,x) \le r\big) = \mathbb{P}_{\mathfrak{d}_x^{-r}}\big(h(t,\cdot) \le -h_0^{[a,b]}\big). \tag{384}$$

The third equality comes from the skew time reversal invariance of PNG (219), with $\mathfrak{d}_x^{-r}$ a shifted narrow wedge given by $\mathfrak{d}_x^{-r}(y) = -r$ if $x = y$ and $-\infty$ otherwise. In view of this we may compute

$$\partial_{b^-} F_{r,a,b}\big|_{b=x+t} = \lim_{\delta\to 0} \delta^{-1}\Big(\mathbb{P}_{\mathfrak{d}_x^{-r}}\big(h(t,\cdot) \le -h_0^{[a,x+t]}\big) - \mathbb{P}_{\mathfrak{d}_x^{-r}}\big(h(t,\cdot) \le -h_0^{[a,x+t-\delta]}\big)\Big) \tag{385}$$

as

$$-\lim_{\delta\to 0} \delta^{-1}\mathbb{P}_{\mathfrak{d}_x^{-r}}\big(\forall y \in [a, x+t-\delta],\ h(t,y) \le -h_0(y);\ \exists z \in (x+t-\delta, x+t], h(t,z) > -h_0(z)\big).$$

The key is that the derivative is being computed at the edge of the forward light cone, and that, in view of the initial condition, $h(t,x+t) = -r$. Suppose first that $x+t$ is not a jump point for $-h_0$ so that, in particular, h_0 is constant, and equal to $h_0(x+t)$, on $[x+t-\delta, x+t]$ if δ is small enough. Then on the event inside the probability we have $h(t,x+t-\delta) \le -h_0(x+t)$ while $h(t,x+t) = -r \le -h_0(x+t)$. Hence, for the event to occur, $h(t,y)$ has to jump up *and then down* in the interval $[x+t-\delta, x+t]$, which has probability $\mathcal{O}(\delta^2)$. The same holds if h_0 has a down jump at $x+t$, because h_0 is upper semi-continuous, so it is still constant on $[x+t-\delta, x+t]$. The only relevant possibility then is that h_0 has an up jump at $x+t$. In this case, and up to terms of order δ^2, the event inside the probability will occur if $h(t,y)$ stays below $-h_0$ on $[a, x+t-\delta]$ and $-r = h(t,x) > -h_0(x+t)$, but we are assuming $r \ge h_0(x+t)$. Thus the forcing terms (383) vanish. □

Finally we comment about boundary conditions. At $t = 0$ we clearly have from the definition of $F_r(t,x)$ that

$$F_r(0,x) = \mathbf{1}_{r \geq h_0(x)}. \tag{386}$$

Since we are dealing with a wave equation, we require an extra piece of data at $t = 0$, which in simple cases is given by

$$\partial_t F_r(0,x) = -\textstyle\sum_y \big((h_0(y) - h_0(y^-))\mathbf{1}_{h_0(y^-) \leq r < h_0(y)} + (h_0(y) - h_0(y^+))\mathbf{1}_{h_0(y^+) \leq r < h_0(y)} \big) \delta_y(x). \tag{387}$$

This comes from the deterministic part of the dynamics: The random part does not contribute to the derivative because a jump at x coming from a nucleation in a time interval of length ϵ has probability of order ϵ^2. From the dynamics of the process, $h(t,x)$ can never be smaller than $r_0(t,x)$ and hence

$$F_r(t,x) = 0, \qquad r < r_0(t,x). \tag{388}$$

This leaves open the fate of $F_{r_0(t,x)}(t,x)$. We clearly have $\frac{F_{r_0+1}F_{r_0-1}}{F_{r_0}^2} = 0$, since $F_{r_0-1} = 0$, and $F_{r_0} > 0$, since if there are no nucleations in the backward light cone of (t,x), then $h(t,x) = r_0(t,x)$, and this has positive probability (note however, that other scenarios can also lead to $h(t,x) = r_0(t,x)$ so F_{r_0} is not trivial to compute.) This would appear to give $\frac{1}{4}(\partial_t^2 - \partial_x^2) \log F_r(t,x) \mid_{r=r_0(t,x)} = -1$, but it is not quite true because F_{r_0} has jumps when r_0 has jumps. These can be worked out in the following way. First of all, $r_0(t,x)$ is computed in an elementary way from the initial data $h_0(x)$. It has discontinuities of the first kind (jumps) on (annihilating) lines of slope either 1 or -1 emerging from the initial jumps of h_0 and at each discontinuity point (t,x) we have $r_0(t^-,x) < r_0(t^+,x)$. It is not hard to see that $F_{r_0(t,x)}(t,x)$ must jump from $F_{r_0(t^-,x)}(t^-,x)$ to $F_{r_0(t^+,x)}(t^-,x)$ as we cross the discontinuity line. In the interior of the regions bounded by the discontinuity lines, we do have $\frac{1}{4}(\partial_t^2 - \partial_x^2) \log F_r(t,x) \mid_{r=r_0(t,x)} = -1$. Because the jump is actually computed using the value of F at t^-, $F_{r_0(t,x)}(t,x)$ can now be computed everywhere.

11. KP is 1:2:3 scaling limit of 2D Toda

The 1:2:3 rescaling means we are interested in $\mathfrak{F}_\epsilon$ defined by

$$F(t,x,r) = \mathfrak{F}_\epsilon(\epsilon^{3/2}t, \epsilon x, \epsilon^{1/2}(r - 2t)). \tag{389}$$

For the derivation, we have to expand $\mathfrak{F}_\epsilon$ in Taylor series even though it really lives on $\epsilon^{1/2}\mathbb{Z}$. The idea is that $\mathfrak{F}_\epsilon$ is essentially the restriction to the lattice of a nice function converging nicely to $\mathfrak{F}$. Justifying such an expansion is very long and we don't attempt it here. Under this scaling, (342) becomes

$$\tfrac{1}{4}(\epsilon^3\partial_t^2 - \epsilon^2 4\partial_{tr}^2 + \epsilon 4\partial_r^2 - \epsilon^2\partial_x^2) \log \mathfrak{F} = \tfrac{\mathfrak{F}(\cdot+\epsilon^{1/2})\mathfrak{F}(\cdot-\epsilon^{1/2})}{\mathfrak{F}^2} - 1, \tag{390}$$

where for notational simplicity we removed the ϵ from the subscript in $\mathfrak{F}_\epsilon$. Expanding (390) out and multiplying by $\mathfrak{F}^2$, we get

$$\epsilon^2 \left[\mathfrak{F}\partial_{tr}^2\mathfrak{F} - \partial_t\mathfrak{F}\,\partial_r\mathfrak{F} + \tfrac{1}{4}\mathfrak{F}\,\partial_x^2\mathfrak{F} - \tfrac{1}{4}(\partial_x\mathfrak{F})^2 + \tfrac{1}{12}\mathfrak{F}\,\partial_r^4\mathfrak{F} - \tfrac{1}{3}\partial_r\mathfrak{F}\,\partial_r^3\mathfrak{F} + \tfrac{1}{4}(\partial_r^2\mathfrak{F})^2 \right] + o(\epsilon^2) = 0. \tag{391}$$

The vanishing of the square brackets is the Hirota form of KP. A simple computation shows that it is equivalent to $\phi = \partial_r^2 \log \mathfrak{F}$ satisfying (327).

This gives the following. A rigorous proof can be obtained by going through the analogue of the proof of Theorem 10.1. Since it is very similar, and can easily be achieved by taking the 1:2:3 limit of each expression of the proof of that previous Theorem, we don't include it here.

Theorem 11.1. *Let $K_{t,x,r}$ be trace class operators acting on $L^2(\mathbb{R}_+)$ and*

$$F(t,x,r) = \det(I-K)_{L^2(\mathbb{R}_+)}. \tag{392}$$

Suppose that $K_{t,x,r}$ satisfies

(1) $\partial_r K = [\partial, K]$,
(2) $\partial_x K = -[\partial^2, K]$,
(3) $\partial_t K = -\frac{1}{3}[\partial^3, K]$.

Then $\phi = \partial_r^2 \log F$ satisfies the KPII equation

$$\partial_t\phi + \tfrac{1}{2}\partial_r\phi^2 + \tfrac{1}{12}\partial_r^3\phi + \tfrac{1}{4}\partial_r^{-1}\partial_x^2\phi = 0. \tag{393}$$

In particular, it holds for the one point distribution $F(t,x,r) = P_{\mathfrak{h}_\circ}(\mathfrak{h}(t,x) \leq r)$ of the KPZ fixed point.

Not surprisingly, KP has the necessary invariance under

$$\phi(t,x,r) \mapsto \alpha^{-2}\phi(\alpha^{-3}t, \alpha^{-2}x, \alpha^{-1}r), \qquad \mathfrak{h}_0(x) \mapsto \alpha^{-1}\mathfrak{h}_0(\alpha^2 x), \tag{394}$$

and this embeds the Tracy-Widom distributions into an invariance principle:

Example 11.2. (Tracy-Widom GUE distribution) *Consider $\mathfrak{h}_0 = \mathfrak{d}_0$, the* narrow wedge initial condition *defined as $\mathfrak{d}_0(0) = 0$ and $\mathfrak{d}_0(x) = -\infty$, $x \neq 0$. With this choice of initial data one has $\mathfrak{h}(t,x) + x^2/t \overset{\text{dist}}{=} t^{1/3}\mathcal{A}(t^{-2/3}x)$ where $\mathcal{A}$ is the Airy$_2$ process, which is stationary in x. From this and the 1:2:3 scaling invariance of* (327)*, it is natural to look for a self-similar solution of the form*

$$\phi^{\text{nw}}(t,x,r) = t^{-2/3}\psi^{\text{nw}}(t^{-1/3}r + t^{-4/3}x^2). \tag{395}$$

This turns (327) *into*

$$(\psi^{\text{nw}})''' + 12\psi^{\text{nw}}(\psi^{\text{nw}})' - 4r(\psi^{\text{nw}})' - 2\psi^{\text{nw}} = 0. \tag{396}$$

The transformation $\psi^{\text{nw}} = -q^2$ takes (396) *into Painlevé II:*

$$q'' = rq + 2q^3. \tag{397}$$

As $r \to -\infty$ the solution is approximately $\phi^{\text{nw}}(t,x,r) \sim -(\frac{r}{2t} + \frac{x^2}{2t^2})$, picking out the Hastings-McLeod solution $q(r) \sim -\text{Ai}(r)$ as $r \to \infty$. Thus we recover

$$F(t,x,r) = F_{\text{GUE}}(t^{-1/3}r + t^{-4/3}x^2) \tag{398}$$

where F_{GUE} is the GUE Tracy-Widom distribution [TW94], usually written in the equivalent form

$$F_{\text{GUE}}(s) = \exp\left\{-\int_s^\infty du\,(u-s)q^2(u)\right\}. \tag{399}$$

Example 11.3. (Tracy-Widom GOE distribution) *If $\mathfrak{h}_0(x) \equiv 0$, the* flat initial condition*, there is no x dependence and* (327) *reduces to KdV. Now we look for a self-similar solution of KdV the form*

$$\phi^{\text{fl}}(t,r) = (t/4)^{-2/3}\psi^{\text{fl}}((t/4)^{-1/3}r) \tag{400}$$

(the extra factor of 1/4 is to coordinate conventions with random matrix theory), obtaining the ordinary differential equation

$$(\psi^{\mathrm{fl}})''' + 12(\psi^{\mathrm{fl}})'\psi^{\mathrm{fl}} - r(\psi^{\mathrm{fl}})' - 2\psi^{\mathrm{fl}} = 0. \tag{401}$$

Miura's transform

$$\psi^{\mathrm{fl}} = \tfrac{1}{2}(q' - q^2) \tag{402}$$

brings this to Painlevé II (397)*, with the same behavior as $r \to -\infty$. So we recover*

$$F(t,x,r) = F_{\mathrm{GOE}}(4^{1/3}t^{-1/3}r) \tag{403}$$

where F_{GOE} is the GOE Tracy-Widom distribution [TW96], usually written in the equivalent form

$$F_{\mathrm{GOE}}(r) = \exp\left\{-\frac{1}{2}\int_r^\infty du\, q(u)\right\} F_{\mathrm{GUE}}(r)^{1/2}. \tag{404}$$

Exercise 11.4. Show that for flat initial data the one point distribution satisfies the classic Toda equation (304),(330) in the case of PNG, and KdV in the case of the fixed point. Find the Hirota form of KdV and show that the former converges to the latter under the 1:2:3 scaling.

Exercise 11.5. In order to try to understand KP one might observe ϕ in the frame of the inviscid solution $\frac{1}{4}(\partial_x\bar{\mathfrak{h}})^2 - \partial_t\bar{\mathfrak{h}} = 0$ of Burgers' equation,

$$\phi(t,x,r) := \bar{\phi}(t,x,r-\bar{\mathfrak{h}}(t,x)). \tag{405}$$

Show that

$$\partial_t\bar{\phi} + \bar{\phi}\,\partial_r\bar{\phi} + \tfrac{1}{12}\partial_r^3\bar{\phi} + \tfrac{1}{4}\partial_r^{-1}\partial_x^2\bar{\phi} - \tfrac{1}{2}\partial_x\bar{\mathfrak{h}}\,\partial_x\bar{\phi} + V\bar{\phi} = 0 \tag{406}$$

with $V = -\frac{1}{4}\partial_x^2\bar{\mathfrak{h}}$ and with initial data $\bar{\phi}(0,x,r) = 0$ for $r \geq 0$ and $-\infty$ for $r < 0$.

Exercise 11.6. Let $\bar{\phi}$ be as in (406) and

$$\check{\phi} = \bar{\phi} - \eta \tag{407}$$

where $\eta = \frac{c}{t}r\mathbf{1}_{r<0}$. Show that $\bar{\phi}(0,x,r) = 0$ and

$$\partial_t\check{\phi} + \tfrac{1}{2}\partial_r\check{\phi}^2 + \tfrac{1}{12}\partial_r^3\check{\phi} + \tfrac{1}{4}\partial_r^{-1}\partial_x^2\check{\phi} + \partial_r(\eta\check{\phi}) - \tfrac{1}{2}\partial_x\bar{\mathfrak{h}}\partial_x\check{\phi} - \tfrac{1}{4}\partial_x^2\bar{\mathfrak{h}}\ \check{\phi} + (\tfrac{c-1}{t} - \tfrac{1}{4}\partial_x^2\bar{\mathfrak{h}})\eta - \tfrac{c}{12t}\delta_0'(r) = 0. \tag{408}$$

One hopes to set things up so that $\check{\phi}$ has good decay at $\pm\infty$. Consider our two basic examples. In the flat case $\bar{\mathfrak{h}} \equiv 0$, and we want to take $c = 1$ to make the second last term drop out. In the narrow wedge case $\bar{\mathfrak{h}} = -\frac{x^2}{t}$ so $\frac{1}{4}\partial_x^2\bar{\mathfrak{h}} = -\frac{1}{2t}$ and we want to take $c = 1/2$. Argue informally that this leads to $\phi(t,r) \sim r/t$, or $\log F(t,r) \sim -\frac{1}{6t}r^3$ as $r \to -\infty$ in the Tracy-Widom GOE case; and $\log F(t,x,r) \sim -\frac{1}{12t}(r + \frac{x^2}{t})^3$ in the Tracy-Widom GUE case (NB. Rigorous proofs are quite difficult here. It is usually done with Riemann-Hilbert methods. There is an alternate method using the stochastic Airy operator.)

Exercise 11.7 (Little research project suggested by P. le Doussal)**.** We know that initial data $-|x|^{1/2}$ is a boundary case between flat and narrow wedge (check this using (216)). It is also true that the hit kernel can be computed exactly in this case (try to solve the necessary heat equation for hitting such a curve by introducing a drift and then scaling. The resulting equation is solvable in terms of parabolic cylinder functions – see NIST site for definitions.) Now the question is, for such initial data, what is the $r \to -\infty$ asymptotics like in the previous problem?

Exercise 11.8. Let h_{nw} be the narrow wedge solution of (21) with $\lambda = \nu = \frac{1}{4}$ and $\sigma = 1$. The KPZ generating function is

$$G_{\mathrm{nw}}(t,x,r) = \mathbb{E}\Big[\exp\Big\{-e^{\mathrm{h}_{\mathrm{nw}}(t,x)+\frac{t}{12}-r}\Big\}\Big]. \tag{409}$$

The distribution of $\mathrm{h}_{\mathrm{nw}}(t,x)$ was computed earlier, with the result that $G_{\mathrm{nw}}(t,x,r) = \det(\mathbf{I}-\mathbf{K})_{L^2(\mathbb{R}^+)}$ with

$$\mathbf{K}(u,v) = \int_{-\infty}^{\infty} dy\, t^{-2/3}\frac{1}{1+e^y}\mathrm{Ai}(t^{-1/3}(u+r-y)+t^{-4/3}x^2)\mathrm{Ai}(t^{-1/3}(v+r-y)+t^{-4/3}x^2).$$

Show that the differential relations (1)–(3) from Thm. 11.1 are satisfied and therefore conclude that $\phi_{\mathrm{nw}} = \partial_r^2 \log G_{\mathrm{nw}}$ satisfies the KP-II equation (327) (NB. This may require some conjugation of the kernel).

The initial condition is $\lim_{t\searrow 0}\phi_{\mathrm{nw}}(t,x,r-\frac{x^2}{t}-\log\sqrt{\pi t}) = -e^{-r}$. This suggests defining the x independent, shifted variable $\hat\phi_{\mathrm{nw}}(t,r) = \phi_{\mathrm{nw}}(t,x,r-\frac{x^2}{t}-\log\sqrt{\pi t})$. Show that this satisfies the *cylindrical KdV equation*,

$$\partial_t\hat\phi_{\mathrm{nw}} + \tfrac{1}{2t}\partial_r\hat\phi_{\mathrm{nw}} + \hat\phi_{\mathrm{nw}}\partial_r\hat\phi_{\mathrm{nw}} + \tfrac{1}{12}\partial_r^3\hat\phi_{\mathrm{nw}} + \tfrac{1}{2t}\hat\phi_{\mathrm{nw}} = 0, \qquad \hat\phi_{\mathrm{nw}}(0,r) = -e^{-r}. \tag{410}$$

Exercise 11.9. Consider now the solution h_b of (21) with $\lambda = \nu = \frac{1}{4}$, $\sigma = 1$, and m-spiked initial data, where $b = (b_1,\cdots,b_m)\in\mathbb{R}^m$ are the spike parameters. When $m=1$, this corresponds to half-Brownian initial data (more precisely, at the level of the SHE one sets $Z(0,x) = e^{B(x)+b_1x}\mathbf{1}_{x\geq 0}$ where $B(x)$ is a Brownian motion with diffusivity 2); for the definition in the general case $m\geq 1$ see [BCF14, Defn. 1.9]. Define G_b as in (409) with h_b in place of h_{nw}. Then from [BCF14, Thm. 1.10] after a few small manipulations,

$$G_b(t,x,r) = \det(\mathbf{I}-\widetilde{\mathbf{K}})_{L^2[0,\infty)} \tag{411}$$

with

$$\widetilde{\mathbf{K}}(u,v) = \frac{1}{(2\pi\mathrm{i})^2}\int_{\mathcal{C}_1} d\xi \int_{\mathcal{C}_1'} d\eta\, \frac{\pi}{\sin(\pi(\xi-\eta))}\frac{e^{t\xi^3/3+x\xi^2-(u+r)\xi}}{e^{t\eta^3/3+x\eta^2-(v+r)\eta}}\prod_{k=1}^{m}\frac{\Gamma(\eta-b_k)}{\Gamma(\xi-b_k)}$$

where $\mathcal{C}_1$ goes from $-\frac14-\mathrm{i}\infty$ to $-\frac14+\mathrm{i}\infty$ crossing the real axis to the right of $b_1,\ldots,b_m$ and $\mathcal{C}_1' = \mathcal{C}_1+\frac12$. As in the previous example show that $\tilde{\mathbf{K}}$ satisfies the necessary differential relations, so that $\phi_b = \partial_r^2\log G_b$ satisfies the KP-II equation (327)

Exercise 11.10. Consider the KPZ equation with two-sided Brownian initial data, i.e. the solution $\mathrm{h}_{w_\pm}$ of (21) with initial data of the form $\mathrm{h}_{w_\pm}(0,x) = \mathbf{B}(x) + w_- x\mathbf{1}_{x<0} + w_+ x\mathbf{1}_{x\geq 0}$ with $\mathbf{B}$ a double-sided Brownian motion with $\mathbf{B}(0)=0$ and $w_- > w_+$. Define

$$\widetilde{G}_{w_\pm}(t,x,r) = \Gamma(w_- - w_+)^{-1}\mathbb{E}\Big[2e^{\frac12(w_--w_+)(\mathrm{h}_{w_\pm}(t,x)+\frac{t}{12}-r)}K_{w_+-w_-}(2e^{\frac12(\mathrm{h}_{w_\pm}(t,x)+\frac{t}{12}-r)})\Big]$$

where K_ν is the modified Bessel function of order ν. This modified generating function $\widetilde{G}_{w_\pm}$ can alternatively be expressed as the analog of (409) where the KPZ height function $\mathrm{h}_{w_\pm}$ is replaced by a randomly shifted height function $\mathrm{h}_{w_\pm}(t,x)+\Upsilon$,

$$\widetilde{G}_{w_\pm}(t,x,r) = \mathbb{E}\Big[\exp\Big\{-e^{\mathrm{h}_{w_\pm}(t,x)+\Upsilon+\frac{t}{12}-r}\Big\}\Big]$$

with Υ an independent log-gamma random variable with parameter $w_- - w_+$, i.e. $e^{-\Upsilon}$ has density $\Gamma(w_- - w_+)^{-1}x^{w_--w_+-1}e^{-x}$. Explicit formulas for the distribution of this shifted height function were obtained in [IS12; IS13] using the non-rigorous replica method. A similar formula, which is the one we will use below, was obtained rigorously later on [Bor+15]; the equality between the two expressions above for $\widetilde{G}_{w\pm}$ is essentially Cor. 2.6 in the latter paper, see also Rem. 2.10 there. From [Bor+15, Thm. 2.9] we have (again after a few minor manipulations)

$$\widetilde{G}_{w\pm}(t,x,r) = \det(\mathbf{I} - \widetilde{\mathbf{K}})_{L^2[0,\infty)} \tag{412}$$

with

$$\widetilde{\mathbf{K}}(u,v) = \frac{1}{(2\pi\mathrm{i})^2}\int_{\mathcal{C}_1} d\eta \int_{\tilde{\mathcal{C}}_1} d\xi\, \frac{\pi}{\sin(\pi(\xi-\eta))} \frac{e^{t\xi^3/3+x\xi^2-(u+r)\xi}}{e^{t\eta^3/3+x\eta^2-(v+r)\eta}} \frac{\Gamma(w_- - \xi)}{\Gamma(\xi - w_+)} \frac{\Gamma(\eta - w_+)}{\Gamma(w_- - \eta)}.$$

As in the previous exercises, show that $\widetilde{\mathbf{K}}$ satisfies the hypotheses of Thm. 11.1, so $\widetilde{\phi}_{w\pm}(t,x,r) = \partial_r^2 \log \widetilde{G}_{w\pm}(t,x,r)$ satisfies the KP-II equation (327).

Exercise 11.11. Consider the function ϕ_{nw} from (410). Show that from the definition as a generating function it has a formal expansion

$$\phi_{\mathrm{nw}}(t,r) = \sum_n \frac{(-1)^n n^2}{n!} Z_n(t) e^{\frac{nt}{12} - nr} \qquad Z_n(t) = E[Z^n(t,0)]. \tag{413}$$

Use Exercise 3.12 to show that the sum is divergent. Inserting (413) into (410) obtain the system

$$-\partial_t Z_n + (\frac{n^3-n}{12} - \frac{1}{2t})Z_n = -\frac{(n-1)!}{2} \sum_{n_1+n_2=n} \frac{n_1^2 n_2^2}{n_1! n_2!} Z_{n_1} Z_{n_2}. \tag{414}$$

Show that $E[Z(t,x)]$ must be given by the heat kernel. Show that this is consistent with the equation for Z_1 in (414). The solution of $Z_2(t)$ by Bethe ansatz is

$$Z_2(t,x) = \tfrac{1}{\sqrt{32\pi t}} e^{t/2}(1 + \mathrm{Erf}(\sqrt{t/2})). \tag{415}$$

Show that this is consistent with the equation for Z_2 in (414).

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