UVA Summer School PS1

July 8th 2024

1. **Exercise 1.1** Let $\mathcal{E}$ be a nice subset of $\mathbb{R}^d$ and e_n, $n = 1, 2, \ldots$ be an orthonormal basis of $L^2(\mathcal{E})$ and $\xi_1, \xi_2, \ldots$ be independent standard Gaussians (i.e. mean zero and variance 1).

   - Show that $\xi = \sum_n \xi_n \mathrm{e}_n$ is Gaussian white noise on $\mathcal{E}$. Why is it called *white*?
   - Show that the space-time white noise rescales as,

   $$\xi(t, x) \stackrel{\text{dist}}{=} \epsilon^{\frac{z+1}{2}} \xi(\epsilon^z t, \epsilon^1 x). \tag{1}$$

   Now start with the KPZ equation,

   $$\partial_t h = \lambda(\partial_x h)^2 + \nu \partial_x^2 h + \sqrt{D}\xi \tag{2}$$

   Rescale it as

   $$h_\epsilon(t, x) = \epsilon^b h(\epsilon^{-z} t, \epsilon^{-1} x) \tag{3}$$

   - Show that

   $$\partial_t h_\epsilon = \epsilon^{2-z-b} \lambda(\partial_x h_\epsilon)^2 + \epsilon^{2-z} \nu \partial_x^2 h_\epsilon + \epsilon^{b - \frac{1}{2}z + \frac{1}{2}} \sqrt{D}\xi. \tag{4}$$

2. **Exercise 4.14** [Two type random walks with creation and mutual annihilation] Consider a random walk with state space $S = (\mathbb{Z}^2_{\geq 0})^{\mathbb{Z}}$. At every sites we have two types of particles η_x and ζ_x. There are two parts of the dynamics. Creation and annihilation in pairs: At rate c, an η and a ζ particle are added to x and at rate $d\eta_x\zeta_x$, one particle of each type is removed from x. Jumping: The η particles jump as continuous time random walks purely to the right, and the ζ particles jump as continous time random walks purely to the left. The generator is $L = L_{\mathrm{cr}} + L_{\mathrm{rw}}$ where

   $$\begin{aligned} L_{\mathrm{cr}} f(\eta, \zeta) &= \textstyle\sum_x c(f(\eta^{+,x}, \zeta^{+,x}) - f(\eta, \zeta)) + d\eta_x\zeta_x(f(\eta^{-,x}, \zeta^{-,x}) - f(\eta, \zeta)) \\ L_{\mathrm{rw}} f(\eta, \zeta) &= \textstyle\sum_x r\eta_x(f(\eta^{x \to x+1}, \zeta) - f(\eta, \zeta)) + r\zeta_x(f(\eta, \zeta^{x \to x-1}) - f(\eta, \zeta)) \end{aligned} \tag{5}$$

   - Show that η_x, ζ_x products of Poissons with parameters ϱ_1, ϱ_2 with $\varrho_1\varrho_2 = c/d$ are invariant for each of L_{cr} and L_{rw}, and therefore for L. Does it even depend on the jump law of the random walks?
   - Show that the adjoint L^* just has $c \leftrightarrow d$ and the directions of the walks of the η and ζ particles reversed.

3. **Exercise 1.12** You may have heard of *viscosity solutions* of Burgers' equation. You may even have been told these are the "true" solutions. There is an intrinsic notion of viscosity solutions, which we will not deal with, but the original idea was that one should add some viscosity, solve the equation, then remove it to find the physical solution. After all, there is supposed to be a tiny bit of friction around always. So we solve

   $$\partial_t h_\epsilon = \lambda(\partial_x h_\epsilon)^2 + \epsilon^{1/2} \nu \partial_x^2 h_\epsilon \tag{6}$$

   and let ϵ go to zero to find our solution.

   - Use the Cole-Hopf transformation to find an explicit solution of (6) in terms of convolution of the initial condition with heat kernels.

- Now take a limit as $\epsilon \to 0$ to obtain the variational formula for the resulting limit

$$h(t,x) = \sup_y \left\{ h(0,y) - \tfrac{(x-y)^2}{4\lambda t} \right\}. \tag{7}$$

If you let $h(0,x) =$ two sided Brownian motion, you can actually compute the resulting $h(t,x)$. It is not easy, it can be found in [FM00] (following [Gro89]). The important point though is that one can easily check from the formula in [FM00] that $h(t,\cdot)$ is *not* a two-sided Brownian motion.

- Now check that the resulting process is 1:2:3 invariant. It is an example of a 1:2:3 invariant process that is not the KPZ fixed point.

4. **Exercise 3.12** Let $z(t,x)$ be the solutions of the unmollified stochastic heat equation.

$$\partial_t z = \frac{1}{2}\partial_x^2 z + \xi z, \quad z(0,x) = z_0(x) \tag{8}$$

Estimate the size of the N-th moments $\mathbb{E}[z^N(t,x)]$ as follows.

- Use Ito's formula on the function[1] $\sum_{i<j} |x_i - x_j|$ to obtain

$$\int_0^t \sum_{i<j} \delta(B_i - B_j) ds = \tfrac{1}{2}\sum_{i<j} |B_i(t) - B_j(t)| - \tfrac{1}{2}\sum_{i<j}\int_0^t \operatorname{sgn}(B_i - B_j)(dB_i - dB_j). \tag{9}$$

So

$$E[z^N(t,x)] = E^{x,\ldots,x}\left[e^{-\frac{1}{2}\sum_{i<j}\int_0^t \operatorname{sgn}(B_i - B_j)(dB_i - dB_j)} F(B(t)) \right] \tag{10}$$

where $F(B) = e^{\frac{1}{2}\sum_{i<j}|B_i - B_j|}\prod_{i=1}^N z_0(B_i)$. Now if z_0 has support in $[-L,L]$ and is bounded by C then $F \le e^{C'N^2}$ and if z_0 is bounded below by $c > 0$ then $F \ge c^N$.

- Use the antisymmetry to write

$$\sum_{i<j}\int_0^t \operatorname{sgn}(B_i - B_j)(dB_i - dB_j) = \sum_{i=1}^N \int_0^t \sum_{j\neq i}\operatorname{sgn}(B_i - B_j) dB_i \tag{11}$$

- Use the Itô isometry to obtain

$$E\left[e^{\frac{1}{2}\sum_{i=1}^N \int_0^t \{\sum_{j\neq i}\operatorname{sgn}(B_i - B_j)\} dB_i} \right] = E\left[e^{\frac{1}{8}\sum_{i=1}^N \int_0^t \{\sum_{j\neq i}\operatorname{sgn}(B_i - B_j)\}^2 ds} \right]. \tag{12}$$

- Show that $\sum_{j\neq i}\operatorname{sgn}(B_i - B_j) = N - 1 - 2(k_i - 1)$ where $1 \le k_i \le N$ is the position of B_i when you order $(B_1,\ldots,B_N)$. The sum over i can be taken in any order so

$$\tfrac{1}{8}\sum_{i=1}^N \{\sum_{j\neq i}\operatorname{sgn}(B_i - B_j)\}^2 = \tfrac{1}{24}N(N^2-1) \tag{13}$$

and doesn't depend on the B_i's at all.

- Conclude that

$$\mathbb{E}[z^N(t,x)] \sim E\left[e^{\frac{1}{8}\sum_{i=1}^N \int_0^t \{\sum_{j\neq i}\operatorname{sgn}(B_i - B_j)\}^2 ds} \right] = e^{\frac{1}{24}N(N^2-1)t}. \tag{14}$$

The tilde is because we threw away the term F by upper or lower bounding it by things much smaller than the key term $e^{\frac{t}{24}N^3}$.

[1] Although this function is technically not allowed for Itô's formula, it is very easy to make sense of these computations by mollifying the function $|x|$ near 0 and then removing the mollification. See any proof of Tanaka's formula for details.