UVA Summer School PS2

July 9th 2024

1. **Exercise 6.1** Starting the PNG from the narrow wedge initial condition $\mathfrak{d}_0$, i.e.

$$h(0,x) = \begin{cases} 0, x = 0 \\ -\infty, x \neq 0 \end{cases}, \tag{1}$$

we will show that the one-point marginal $h(t, 0)$ is equivalent to a Poissonized version of the longest increasing subsequence problem[1].

The longest increasing subsequence problem is as follows: consider the problem of finding Lipschitz-1 paths going from $(0, 0)$ to $(t, 0)$ which pick up a maximal number of space-time Poisson points along the way. All these paths lie inside the square R with vertices $(0, 0)$, $(t/2, t/2)$, $(t/2, -t/2)$ and $(t, 0)$, and

- Show that the maximal number of points which they can pick up is exactly $h(t, 0)$.

Now rotate the picture by $-45°$, let N denote the number of Poisson points inside the square corresponding to R (so that N is a Poisson$[t]$ random variable), and order these N points according to their x coordinate. The y coordinates of these points define a random permutation σ of $\{1, \ldots, N\}$, which is clearly chosen uniformly from S_N.

- Show that $h(t, 0)$ is nothing but the length of the longest increasing subsequence in σ.

2. **Exercise 6.8** Recall some notions from the lecture. $E_{a,u}$ is expectation of the height function, which jumps down for η particles and up for ζ particles, i.e. $h_x - h_{x-1} = \zeta_x - \eta_x$ with $h_a = u$ with respect to the product measure from Exercise 4.14 (problem 2 from Monday) on the interval $\{a, a+1, \ldots, b\}$. Note that we are dealing with the conditional measure that $h(b) = v$, multiplied by the probability that $h(b) = v$, so we can write it as $E_{a,u}[(L^* F)\mathbf{1}_{h_b=v}]$, where $L^* = L^*_{\mathrm{rw}} + L^*_{\mathrm{cr}}$. $F(g)$ is the indicator function that $g - h$ is ever less than or equal to 0.

- Show that $E_{a,u}[(L^*_{\mathrm{cr}} F)\mathbf{1}_{h_b=v}] = 0$.

3. **Exercise 4.6** Let Δ be the discrete Laplace operator on $\mathbb{Z}$, i.e.$\Delta f(x) = f(x+1) - 2f(x) + f(x-1), x \in \mathbb{Z}$.

- Show that

$$e^{x\Delta}(u_1, u_2) = e^{-2x} I_{|u_2-u_1|}(2x), \tag{2}$$

where $I_n(2x) := \frac{1}{2\pi i} \oint_{\gamma_0} dz\, e^{x(z+z^{-1})}/z^{n+1}$ is the modified Bessel function of the first kind. Here γ_0 is any simple positively oriented contour around $0 \in \mathbb{C}$.

[1] This was the first exact fluctuation found in the KPZ class, by Baik, Deift in Johansson in 1999. They took a limit of a determinantal formula for the distribution which derived by Gessel about 10 years earlier (it took the decade to realize that it was a formula for this model) and obtained in the limit the same GUE Tracy-Widom distribution which had also been discovered, again about 10 years earlier, for the asymptotic distribution of the top eigenvalue of a matrix from the Gaussian Unitary Ensemble.

4. **Exercise 6.17** [Narrow wedge initial data] We are going to work out the distribution for the PNG starting from narrow wedge initial data, which is:

$$F_\tau(s) = \mathbb{P}(h(t,x;\mathfrak{d}_0) \le r) = \det(I - K^{\mathfrak{d}_0}(t,x,r))_{l(\mathbb{Z}_{>0})}. \tag{3}$$

The kernel $K^{\mathfrak{d}_0}$ is defined as follows:

$$K^{\mathfrak{d}_0} = e^{-2t\nabla - x\Delta} T^{\mathfrak{d}_0}_{x-t,x+t} e^{2t\nabla + x\Delta} \tag{4}$$

where $T^{\mathfrak{d}_0}_{a,b} = e^{a\Delta} P^{hit(\mathfrak{d}_0)}_{a,b} e^{-b\Delta}$. Recall $P^{hit(\mathfrak{d}_0)}_{a,b}$ is defined as

$$P^{\text{hit}(h)}_{a,b}(u,v) = P(g \text{ hits the hypograph of } h \text{ on } [a,b] \text{ and } g(b) = v \mid g(a) = u). \tag{5}$$

Here g is a random walk which is the difference of two Poisson process with rate 1.
Since the random walk g can only hit the hypograph of $\mathfrak{d}_0$ at the origin,

- Show that
$$e^{(x-t)\Delta} P^{\text{hit}(\mathfrak{d}_0)}_{x-t,x+t} e^{(-t-x)\Delta} = \bar{\chi}_0, \tag{6}$$
where $\overline{\chi}_0(u) = 1_{u\le 0}$.
- Following the previous exercise, show that
$$e^{2t\nabla + x\Delta}(u_1,u_2) = e^{-2x} \frac{1}{2\pi i} \oint_{\gamma_0} \frac{dz}{z^{u_2-u_1+1}} e^{t(z-z^{-1})+x(z+z^{-1})}, \tag{7}$$
where γ_0 is any simple, positively oriented contour around the origin.

When $t > |x|$ this kernel can be expressed in terms of the Bessel function of the first kind, $J_n(x) = \frac{1}{2\pi i}\oint_{\gamma_0} dz\, e^{x(z-z^{-1})/2}/z^{n+1}$, we have:

$$e^{2t\nabla + x\Delta}(u_1,u_2) = e^{-2x} \left(\tfrac{t-x}{t+x}\right)^{(u_2-u_1)/2} J_{u_2-u_1}\big(2\sqrt{t^2-x^2}\big). \tag{8}$$

- Use this to show that
$$K(u,v) = \left(\tfrac{t+x}{t-x}\right)^{(u-v)/2} B_s(u,v), \qquad B_s(u,v) = \sum_{\ell\le 0} J_{u-\ell}(2s) J_{v-\ell}(2s)$$
with $s = \sqrt{t^2-x^2}$.

B_s is called the *discrete Bessel kernel.*

- Show that it is an *integrable* kernel:
$$B_s(u,v) = \frac{s}{u-v}(J_{u-1}(2s)J_v(2s) - J_u(2s)J_{v-1}(2s)).$$
In order to show this, you need the following recurrence relation on J_n:
$$\frac{n}{t} J_n(2t) = J_{n+1}(2t) + J_{n-1}(2t).$$
- Show that a conjugation of kernel reduces the distribution to be:
$$F_r(s) = \det(I - \tau_r B_s \tau_{-r})_{\ell^2(\mathbb{Z}_{>0})}, \tag{9}$$
where $\tau_r f(u) = f(u+r)$.

Note that from [Borodin-Okounkov 2000],

$$\det(I - \tau_r B_s \tau_{-r})_{\ell^2(\mathbb{Z}_{>0})} = e^{-s^2} \det(I_{i-j}(2s))_{i,j=0,\dots,r-1}, \tag{10}$$

where the I_n are modified Bessel functions of the first kind. This latter is Gessel's formula.