# UVA Summer School PS3

July 11th 2024

**Exercise 7.1**[Flat initial data] Let $\mathfrak{h}_0 = 0$ be the initial condition for the KPZ fixed point. Recall the Brownian scattering transform is defined to be:

$$\lim_{L\to\infty} e^{-t\partial^3/3 - L\partial^2} P^{\mathrm{hit}(h)}_{-L,L} e^{t\partial^3/3 - L\partial^2}$$

for some $t > 0$ where $P^{\mathrm{hit}(h)}_{-L,L}(u,v)dv$ is the probability a Brownian motion starting at $u$ at time $-L$ is in $[v, v+dv]$ at time $L$ and hits $h$ along the way.

- Use the reflection principle to show that as $t \searrow 0$ this is the reflection operator.
- Use the formula from exercise (5.8) to compute the finite dimensional distributions of the $\mathrm{Airy}_1$ process in $x$ (= the KPZ fixed point at time 1 starting from flat, $h(1, x; h_0 = 0)$)

**Exercise 5.8** [ScAiry (Scaled Airy) formula] Recall Airy function $\mathrm{Ai}(x)$ is defined as

$$\mathrm{Ai}(z) = \frac{1}{2\pi i}\int_\Gamma dw e^{\frac{w^3}{3} - zw}$$

where the contour $\Gamma$ starts at the point at infinity with argument $-\pi/3$, connects to 0, and ends at the point at infinity with argument $\pi/3$.

- Use the contour integral formula for the Airy function to prove that

$$\int_{-\infty}^{\infty} dx \mathrm{Ai}(a+x)\mathrm{Ai}(b-x) = 2^{-1/3}\mathrm{Ai}(2^{-1/3}(a+b)) \tag{1}$$

**Exercise 7.3** Let $\mathfrak{h}(t, x; \mathfrak{h}_0)$ denote the KPZ fixed point with initial data $\mathfrak{h}_0$. The *KPZ fixed point* is the height function valued Markov process taking with transition probabilities given by:

$$P_{\mathfrak{h}_0}(\mathfrak{h}(t, x_1) \le r_1, \dots, \mathfrak{h}(t, x_n) \le r_n) = \det\big(I - \mathfrak{K}^{\mathrm{ext}}(t, \vec{x}, \vec{r}, \mathfrak{h}_0)\big)_{L^2(\mathbb{R}_+)^n}. \tag{2}$$

The kernel is

$$\mathfrak{K}^{\mathrm{ext}}_{ij}(t, \vec{x}, \vec{r}, \mathfrak{h}) = -e^{(x_j - x_i)\partial^2 + (r_i - r_j)\partial} 1_{x_i < x_j} + (\mathfrak{S}_{t, x_i, r_i} - \mathfrak{S}^{\mathfrak{h}^-}_{t, x_i, r_i})^* (\mathfrak{S}_{t, -x_j, r_j} - \mathfrak{S}^{\mathfrak{h}^+}_{t, -x_j, r_j}) \tag{3}$$

where

$$\mathfrak{S}^{\mathfrak{h}}_{t,x,r}(z, v) = E_z[\mathfrak{S}_{t, x - \tau_{\mathfrak{h}}, r}(B(\tau), v)], \qquad \mathfrak{S}_{t,x,r} = e^{\frac{t}{3}\partial^3 + x\partial^2 - r\partial}. \tag{4}$$

The expectation is taken with respect to a Brownian motion starting from $z$ and $\tau_{\mathfrak{h}}$ is the hitting time of $\mathfrak{h}$ of a Brownian motion starting at $z$.
Conjecturally, the KPZ fixed point is the unique process with the following properties:

1. *(1:2:3 invariance)* $\alpha\mathfrak{h}(\alpha^{-3}t, \alpha^{-2}x; \alpha^{-1}\mathfrak{h}_0(\alpha^2 x)) \overset{\mathrm{dist}}{=} \mathfrak{h}(t, x; \mathfrak{h}_0)$,$\alpha > 0$.
2. *(Invariance of Brownian motion)* If $B(x)$ is a two-sided Brownian motion with diffusion coefficient 2, then for each $t > 0$, $\mathfrak{h}(t, x; B) - \mathfrak{h}(t, 0; B)$ is also two-sided Brownian motion in $x$ with diffusion coefficient 2.

3. *(Skew time reversibility)* $P\big(\mathfrak{h}(t,x;\mathfrak{g}) \le -\mathfrak{f}(x)\big) = P\big(\mathfrak{h}(t,x;\mathfrak{f}) \le -\mathfrak{g}(x)\big)$.

4. *(Stationarity in space)* $\mathfrak{h}(t,x+u;\mathfrak{h}_0(x-u)) \overset{\text{dist}}{=} \mathfrak{h}(t,x;\mathfrak{h}_0)$.

5. *(Reflection invariance)* $\mathfrak{h}(t,-x;\mathfrak{h}_0(-x)) \overset{\text{dist}}{=} \mathfrak{h}(t,x;\mathfrak{h}_0)$.

6. *(Affine invariance)* $\mathfrak{h}(t,x;\mathfrak{h}_0(x)+a+cx) \overset{\text{dist}}{=} \mathfrak{h}(t,x+\frac{1}{2}ct;\mathfrak{h}_0(x)) + a + cx + \frac{1}{4}c^2t$.

- Prove (2)-(5) by passing to the limit from PNG, and (1) and (6) directly from the transition probability formula.

**Exercise 7.8** Recall $\mathcal{A}(x)$ is the *Airy process*, defined as

$$\mathcal{A}(x) = \mathfrak{h}(1,x;\mathfrak{d}_0) + x^2. \tag{5}$$

Recall $F_{\text{GOE}}(s) = \det(I - P_s B_0 P_s)_{L^2(\mathbb{R},dx)}$ where $B_0(x,y) = \text{Ai}(x+y)$.

- Prove *Johansson's formula*,

$$F_{\text{GOE}}(2^{-1/3}r) = \sup_y \left\{\mathcal{A}(y) - y^2\right\}. \tag{6}$$

**Exercise 8.2** Consider the following special discretization on the KdV equation:

$$\partial_t \phi_n = (\phi_{n+1} + \phi_n + \phi_{n-1})(\phi_{n+1} - \phi_{n-1}) - (\phi_{n+2} - 2\phi_{n+1} + 2\phi_{n-1} - \phi_{n-2}). \tag{7}$$

- Show that the measure with $\phi_n$ i.i.d. $N(0,\sigma^2)$ is invariant for (7).
- Show that for the correct value of $\sigma$ it is also invariant for the very similar equation

$$d\phi_n = [(\phi_{n+1} + \phi_n + \phi_{n-1})(\phi_{n+1} - \phi_{n-1}) + (\phi_{n+1} - 2\phi_n + \phi_n)]dt + dB_{n+1} - dB_n, \tag{8}$$

  where $B_n$ are independent Brownian motions.

- Define $h_{n+1} - h_n = \phi(n)$. Why is $h$ a discretization of the KPZ equation?

It is referred to as the *Sasamoto-Spohn model*. Note that neither (7) nor (8) is expected to be integrable in any sense.