UVA Summer School PS4

July 12th 2024

Exercise 1. The one point distribution of PNG, $F(t,x,r) = P(h(t,x;h_0) \leq r)$ satisfy the 2D Toda equation

$$\partial_\eta \partial_\zeta \log F = \frac{F_{r+1}F_{r-1}}{F_r^2} - 1. \tag{1}$$

for $r > r_0(t,x) := \sup_{|y-x|\leq t} h_0(y)$, where $\eta = \frac{1}{2}(t+x), \zeta = \frac{1}{2}(t-x)$

The classic 1d Toda lattice is

$$\ddot{g}_r = e^{g_{r+1}-g_r} - e^{g_r - g_{r-1}}. \tag{2}$$

- In the flat case $h_0 \equiv 0$, find the physical quantity for PNG which solves the classic 1d Toda lattice.

Exercise 2. The one point distribution of the KPZ fixed point , $F(t,x,r) = P(\mathfrak{h}(t,x;\mathfrak{h}_0) \leq r)$ satisfies the following: $\phi = \partial_r^2 \log F$ is a solution of the KP equation

$$\partial_t \phi + \tfrac{1}{2}\partial_r \phi^2 + \tfrac{1}{12}\partial_r^3 \phi + \tfrac{1}{4}\partial_r^{-1}\partial_x^2 \phi = 0. \tag{3}$$

Consider $\mathfrak{h}_0 = \mathfrak{d}_0$, the *narrow wedge initial condition* defined as $\mathfrak{d}_0(0) = 0$ and $\mathfrak{d}_0(x) = -\infty$, $x \neq 0$. With this choice of initial data one has $\mathfrak{h}(t,x) + x^2/t \stackrel{\text{dist}}{=} t^{1/3}\mathcal{A}(t^{-2/3}x)$ where $\mathcal{A}$ is the Airy$_2$ process, which is stationary in x. From this and the 1:2:3 scaling invariance of the KP, it is natural to look for a self-similar solution of the form

$$\phi(t,x,r) = t^{-2/3}\psi(t^{-1/3}r + t^{-4/3}x^2). \tag{4}$$

- Show that this turns (3) into

$$(\psi)''' + 12\psi(\psi)' - 4r(\psi)' - 2\psi = 0. \tag{5}$$

- Show that a solution is given by $\psi = -q^2$ where q solves Painlevé II:

$$q'' = rq + 2q^3. \tag{6}$$