On planar Brownian motion singularly tilted through a point potential

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July 8-19, 2024

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Contents

Motivation

• Our motivation for this study was to understand the correlation measure for a critical two-dimensional random continuum polymer measure corresponding to the 2d SHF (which is our ongoing research topic).

• The *critical two-dimensional stochastic heat flow* (2d SHF) is a distributional limit of point-to-point partition functions for (1+2)-dimensional models for a directed polymer in a random environment (DPRE) within a critical weak-coupling scaling regime.

• In this work, we have focused on constructing the process and investigating its important properties.

Background Material

Brownian local time

- Let $\mathbf{B}$ be a one-dimensional Brownian motion on $(\Omega, \mathscr{F}, \mathbb{P})$.

- For $\omega \in \Omega$, the zero set of $\mathbf{B}$ is defined as

$$\mathscr{O}(\omega) = \big\{t \in (0, \infty) \ : \ \mathbf{B}_t(\omega) = 0\big\}.$$

- How much time does the Brownian motion $\mathbf{B}$ spend at the origin?

- The zero set is closed, uncountable, and has Lebesgue measure zero almost surely.

- Paul Lévy introduced the notion of local time, which is the most natural tool to measure the size of the zero set $\mathscr{O}(\omega)$.

Brownian local time, cont.

- Let $\mathbf{B}$ be a one-dimensional Brownian motion on $(\Omega, \mathscr{F}, \mathbb{P})$.
- Given $\varepsilon \in (0,1)$, define the random variable

$$\mathbf{L}_t^{\varepsilon} = \frac{1}{2\varepsilon} \underbrace{\text{meas}\Big\{s \in [0,t] \;:\; |\mathbf{B}_s| \leq \varepsilon\Big\}}_{\text{Time spent by } \mathbf{B} \text{ within distance } \leq \varepsilon},$$

where $\text{meas}(S)$ denotes the Lebesgue measure of a set $S \subset \mathbb{R}$.

- There exists a continuous process $\{\mathbf{L}_t\}_{t\in[0,\infty)}$ such that

$$\mathbf{L}_t^{\varepsilon} \quad \xrightarrow[\varepsilon\to 0]{\mathbb{P}} \quad \mathbf{L}_t.$$

- The process $\mathbf{L}$ is called the *local time* of the Brownian motion $\mathbf{B}$ at the origin.

Behavior of local time

• Let $\vartheta(\omega, \cdot)$ denote the Borel measure on $[0, \infty)$ with distribution function $t \mapsto \mathbf{L}_t(\omega)$, i.e., for all $0 \leq s < t$

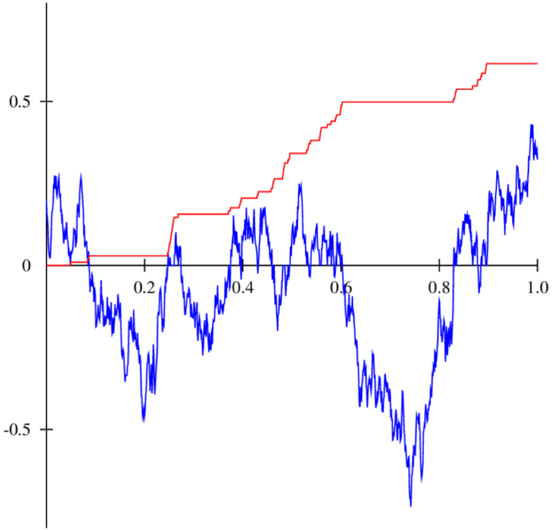
$$\vartheta\big(\omega, (s, t]\big) := \mathbf{L}_t(\omega) - \mathbf{L}_s(\omega).$$

• Then $\vartheta(\omega, \cdot)$ is almost surely supported on the zero set $\mathscr{O}(\omega)$, i.e.,

$$\vartheta\Big(\omega, \big(\mathscr{O}(\omega)\big)^c\Big) = 0.$$

Heuristically, this means that the local time process $\mathbf{L}_t(\omega)$ increases only when $\mathbf{B}_t(\omega) = 0$.

Visualizing the sample paths of Local time



(The sample paths of local time look like the Cantor function.)

Two-dimensional analog of local time?

Is there an analog of Brownian local time at the origin (or any fixed point) in the dimension two case?

• A two-dimensional Brownian motion **B** does not visit the origin after time $t = 0$, i.e.,

$$\mathscr{O}(\omega) := \{t \in (0, \infty) : \mathbf{B}_t(\omega) = 0\} = \emptyset \text{ a.s.}$$

• Consequently, there is not an analogous two-dimensional Brownian local time (at the origin).

$$\mathbf{L}_T = 0$$

Model Formulation

An integral kernel arising in several articles

We will discuss an $\mathbb{R}^2$-valued Markov process defined through an interesting integral kernel f_t^λ arising in several articles:

Bertini and Cancrini
The two-dimensional stochastic heat equation: renormalizing a multiplicative noise
J. Phys. A: Math. Gen. (1998)

Gu, Quastel, Tsai
Moments of the 2d SHE at criticality
Prob. Math. Phys. (2021)

Caravenna, Sun, Zygouras
The critical 2d stochastic heat flow
Inventiones mathematicae. (2023)

Y.-T. Chen
The critical 2d delta-Bose gas as mixed-order asymptotics of planar Brownian motion
arXiv:2105.05154 (2021)

The integral kernel and special functions

- For $\lambda, t > 0$ define $f_t^\lambda : \mathbb{R}^2 \times \mathbb{R}^2 \to (0, \infty)$ by

$$f_t^\lambda(x, y) \, = \, g_t(x - y) \, + \, h_t^\lambda(x, y) \, ,$$

where $g_t(x) := \frac{1}{2\pi t} e^{-\frac{1}{2t}|x|^2}$ and $h_t^\lambda : \mathbb{R}^2 \times \mathbb{R}^2 \to [0, \infty]$ is given by

$$h_t^\lambda(x, y) := 2\pi\lambda \int_{0<r<s<t} g_r(x) \, \nu'\big((s - r)\lambda\big) \, g_{t-s}(y) \, ds \, dr \, ,$$

wherein ν' is the derivative of the *Volterra function*

$$\nu(x) = \int_0^\infty \frac{x^s}{\Gamma(s + 1)} ds \, .$$

- Next, we define $H_t^\lambda : \mathbb{R}^2 \to [0, \infty]$ as the partial integral of $h_t^\lambda(x, \cdot)$.

Different representations of the function $f_t^{\lambda}(x,y)$

Recall that

$$f_t^{\lambda}(x,y) = g_t(x-y) + 2\pi\lambda \int_{0<r<s<t} g_r(x)\, \nu'\big((s-r)\lambda\big)\, g_{t-s}(y)\, ds\, dr\,,$$

- The function f_t^{λ} has the following form in **Gu, Quastel, Tsai**, *Moments of the 2d SHE at criticality*, Prob. Math. Phys. (2021)

$$\begin{aligned}&\big(P + D^{\mathrm{Dgm}(2)}\big)(t, x_1, x_2, x_1', x_2') \\ &= \mathsf{p}\big(\tfrac{1}{2}t, x_{\mathrm{c}} - x_{\mathrm{c}}'\big)\Big(\mathsf{p}\big(2t, x_{\mathrm{d}} - x_{\mathrm{d}}'\big) + \int_{\tau_0+\tau_{1/2}+\tau_1=t} \mathrm{d}\vec{\tau}\, \mathsf{p}\big(2\tau_0, x_{\mathrm{d}}\big)\, 4\pi\mathsf{j}\big(\tau_{1/2}, \beta_\star\big)\, \mathsf{p}\big(2\tau_1, x_{\mathrm{d}}'\big)\Big),\end{aligned} \tag{2.7}$$

Connecting $f_t^\lambda(x,y)$ to a $2d$ Schrödinger point potential

For each $\lambda > 0$ there is a Schrödinger Hamiltonian $\mathbf{H}^\lambda$ on $L^2(\mathbb{R}^2)$ that can be understood as acting on $\psi \in H^{2,2}(\mathbb{R}^2 \backslash \{0\})$ as

$$\mathbf{H}^\lambda \psi(x) = -\frac{1}{2}\Delta\psi(x) = -\frac{1}{2}\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right)\psi(x) \ , \qquad x \in \mathbb{R}^2 \backslash \{0\} ,$$

with the following asymptotic boundary condition near $x = 0$

$$\psi(x) \overset{|x| \ll 1}{\sim} \left(\substack{\text{constant} \\ \text{depending on } \psi}\right) \cdot \left(\log \frac{\lambda |x|^2}{2} + \gamma_{\text{EM}}\right) + o(1) ,$$

where $\gamma_{\text{EM}} = .577\cdots$ is the Euler-Mascheroni constant.

(See *Solvable Models in Quantum Mechanics* by Albeverio et al. pp. 97-98)

Then $f_t^\lambda(x,y)$ is the integral kernel of $e^{t\mathbf{H}^\lambda}$:

$$\left(e^{t\mathbf{H}^\lambda}\psi\right)(x) = \int_{\mathbb{R}^2} f_t^\lambda(x,y)\, \psi(y)\, dy \ .$$

Defining a transition probability semigroup

Fix $T, \lambda > 0$.

For $0 \le s < t \le T$, define $d_{s,t}^{T,\lambda} : \mathbb{R}^2 \times \mathbb{R}^2 \to [0, \infty]$ for $x, y \in \mathbb{R}^2$ with $x \neq 0$ by

$$d_{s,t}^{T,\lambda}(x,y) \;=\; f_{t-s}^{\lambda}(x,y)\,\frac{1 + H_{T-t}^{\lambda}(y)}{1 + H_{T-s}^{\lambda}(x)}\,.$$

Then

$$\int_{\mathbb{R}^2} d_{s,t}^{T,\lambda}(x,y)\,dy \;=\; 1\,,$$

and $d_{s,t}^{T,\lambda}$ satisfies the Chapman-Kolmogorov relation below holds:

$$\int_{\mathbb{R}^2} d_{r,s}^{T,\lambda}(x,y)\,d_{s,t}^{T,\lambda}(y,z)\,dy \;=\; d_{r,t}^{T,\lambda}(x,z)\,, \qquad r < s < t\,.$$

A forward Kolmogorov equation, drift, and an SDE

- $d_{s,\cdot}^{T,\lambda}(x,\cdot)$ satisfies the $2d$ forward Kolmogorov equation

$$\frac{\partial}{\partial t}\left[d_{s,t}^{T,\lambda}(x,y)\right] = \frac{1}{2}\Delta_y\, d_{s,t}^{T,\lambda}(x,y) - \nabla_y \cdot \left[b_{T-t}^{\lambda}(y)\, d_{s,t}^{T,\lambda}(x,y)\right],$$

for $\Delta_y := \frac{\partial^2}{\partial y_1^2} + \frac{\partial^2}{\partial y_2^2}$, $\nabla_y := \left(\frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}\right)$, and the drift $b_t^{\lambda} : \mathbb{R}^2 \to \mathbb{R}^2$ is

$$b_t^{\lambda}(y) := \nabla_y \log\left(1+H_t^{\lambda}(y)\right) = -\frac{y}{|y|}\mathbf{b}_t^{\lambda}(|y|),$$

for $y \neq 0$, where $\mathbf{b}_t^{\lambda} : (0,\infty) \to (0,\infty)$ is a decreasing function.

- The corresponding stochastic differential equation has the form

$$d\mathbf{X}_t = d\mathbf{B}_t + b_{T-t}^{\lambda}(\mathbf{X}_t)\, dt,$$

where $\{\mathbf{B}_t\}_{t\in[0,T]}$ is a standard two-dimensional Brownian motion.

The radial process

The radial process $\mathbf{R}_t := |\mathbf{X}_t|$ satisfies the SDE

$$\underbrace{d\mathbf{R}_t = d\overline{\mathbf{B}}_t + \frac{1}{2\mathbf{R}_t}\,dt}\; - \mathbf{b}^{\lambda}_{T-t}(\mathbf{R}_t)\,dt\,,$$

where $\overline{\mathbf{B}}$ is a standard 1d Brownian motion.

• The bracketed equation is the SDE for a dimension-2 Bessel process, which a.s. never returns to the origin.

• When $0 < |\mathbf{R}_t| \ll 1$, $-\mathbf{b}^{\lambda}_{T-t}(\mathbf{R}_t)$ is small compared to $\frac{1}{2\mathbf{R}_t}$:

$$\mathbf{b}^{\lambda}_{T-t}(a) \overset{a\to 0}{\sim} \frac{1}{a\log\frac{1}{a}} \overset{a\to 0}{\ll} \frac{1}{2a}\,.$$

• Does the process $\mathbf{X}$ visit the origin with positive probability? Yes!

Recall: A planar Brownian motion a.s. never returns to the origin.

Path measures

- Define the path space $\mathbf{\Omega} := C\big([0,\infty),\mathbb{R}^2\big)$.
- Let $\mathscr{B}(\mathbf{\Omega})$ denote the Borel σ-algebra on $\mathbf{\Omega}$.
- Let $\mathbf{X} = \{\mathbf{X}_t\}_{t\in[0,\infty)}$ denote the coordinate process on $\mathbf{\Omega}$, i.e.,

$$\mathbf{X}_t(\omega) := \omega(t)\,, \qquad \omega \in \mathbf{\Omega}\,.$$

- Let $\mathcal{F}^{\mathbf{X}} = \{\mathcal{F}^{\mathbf{X}}_t\}_{t\in[0,\infty)}$ be the filtration generated by $\mathbf{X}$, i.e.,

$$\mathcal{F}^{\mathbf{X}}_t := \sigma\big\{\mathbf{X}_s \ : \ s \in [0,t]\big\}\,.$$

Proposition (Clark and M. (2023+))

Fix $x \in \mathbb{R}^2$ and $T, \lambda > 0$. There exists a unique probability measure $\mathbf{P}^{T,\lambda}_x$ on $(\mathbf{\Omega}, \mathscr{B}(\mathbf{\Omega}))$ under which the coordinate process $\mathbf{X}$ has initial distribution δ_x and is Markov with transition density function $d^{T,\lambda}_{s,t}(y,z)$ (with respect to $\mathcal{F}^{\mathbf{X}}$).

Main Results

Reachability of the origin

For $\omega \in \mathbf{\Omega}$, define the zero set of $\mathbf{X}$ as

$$\mathscr{O}(\omega) := \{t \in [0,\infty) : \mathbf{X}_t(\omega) = 0\},$$

and the event $\mathcal{O} = \{\omega : \mathscr{O}(\omega) \neq \emptyset\}$.

Proposition (Clark and M. (2023+))

Fix any $T, \lambda > 0$ and $x \in \mathbb{R}^2 \backslash \{0\}$.

- $\mathbf{P}_x^{T,\lambda}[\mathcal{O}] > 0$.
- $\{\mathbf{X}_t\}_{t\in[0,\infty)}$ *is a two-dimensional Brownian motion under* $\mathbf{P}_x^{T,\lambda}$ *conditioned on the event* $\mathcal{O}^c$.

The local time process at the origin

Given $\varepsilon \in (0,1)$, define the process $\{\mathbf{L}_t^\varepsilon\}_{t\in[0,\infty)}$ by

$$\mathbf{L}_t^\varepsilon := \frac{\log 2}{2\varepsilon^2 \log^2 \frac{1}{\varepsilon}} \underbrace{\text{meas}\Big\{ \{ r \in [0,t] \, : \, |\mathbf{X}_r| \leq \varepsilon \} \Big\}}_{\text{Time spent by } \mathbf{X} \text{ within distance } \leq \varepsilon}.$$

Theorem (Clark and M. (2023+))

Fix some $T, \lambda > 0$ and a Borel measure μ on $\mathbb{R}^2$. There exists a continuous process $\{\mathbf{L}_t\}_{t\in[0,\infty)}$ on $\mathbf{\Omega}$ for which

$$\sup_{t\in[0,T]} \left| \mathbf{L}_t^\varepsilon - \mathbf{L}_t \right| \quad \stackrel{\varepsilon\to 0}{\longrightarrow} \quad 0 \quad \text{in } L^1\big(\mathbf{P}_\mu^{T,\lambda}\big)\text{-norm}.$$

Moreover, $\mathbf{L}$ is $\mathbf{P}_\mu^{T,\lambda}$ almost surely constant over the interval $[T,\infty)$.

The process $\mathbf{L}$ is called the *local time* of $\mathbf{X}$ at the origin.

Recall: The local time does NOT exist for planar Brownian motion.

Radon-Nikodym derivative of $\mathbf{P}_\mu^{T,\lambda'}$ w.r.t $\mathbf{P}_\mu^{T,\lambda}$

Given $T, \lambda, \lambda' > 0$ define $R_T^{\lambda,\lambda'} : \mathbb{R}^2 \to [0,\infty)$ by

$$R_T^{\lambda,\lambda'}(x) := \begin{cases} \frac{1+H_T^{\lambda'}(x)}{1+H_T^{\lambda}(x)} & \text{for } x \neq 0 \\ \lim_{x\to 0} R_T^{\lambda,\lambda'}(x) = \frac{\nu(T\lambda')}{\nu(T\lambda)} & \text{for } x = 0. \end{cases}$$

Theorem (Clark and M. (2023+))

Fix some $T, \lambda, \lambda' > 0$ and a Borel probability measure μ on $\mathbb{R}^2$. The probability measure $\mathbf{P}_\mu^{T,\lambda'}$ is absolutely continuous with respect to $\mathbf{P}_\mu^{T,\lambda}$ and has Radon-Nikodym derivative

$$\frac{d\mathbf{P}_\mu^{T,\lambda'}}{d\mathbf{P}_\mu^{T,\lambda}} = R_T^{\lambda',\lambda}(X_0)\left(\frac{\lambda'}{\lambda}\right)^{\mathbf{L}_T} \quad a.s.\ \mathbf{P}_\mu^{T,\lambda}.$$

Set of times X visits the origin

Proposition (Clark and M. (2023+))

Fix some $T, \lambda > 0$ and a Borel probability measure μ on $\mathbb{R}^2$. Let $\vartheta \equiv \vartheta(\omega, \cdot)$ denote the random Borel measure on $[0, \infty)$ having distribution function $t \mapsto \mathbf{L}_t(\omega)$. The following statements hold for $\mathbf{P}_{\mu}^{T,\lambda}$ almost every $\omega \in \mathbf{\Omega}$.

- *The zero set $\mathscr{O}(\omega)$ is uncountable when $\omega \in \mathcal{O}$.*
- *The set $\mathscr{O}(\omega)$ has Hausdorff dimension 0.*
- *The measure $\vartheta(\omega, \cdot)$ takes full weight on $\mathscr{O}(\omega)$. (i.e., $\mathbf{L}_t$ increases only when $\mathbf{X}_t = 0$)*

The zero set of one-dim Brownian motion has Hausdorff dimension $\frac{1}{2}$.

Summary

• Starting with a special integral kernel arising in several recent articles, we constructed a transition density function for a two-dimensional diffusion process gently attracted to zero.

• We discussed the zero set and local time for this two-dimensional diffusion, emphasizing that these do not exist for Brownian motion in dimension 2.

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The End

Thanks for listening!