Perfect t-embeddings of Uniform Hexagons

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(joint work with T. Berggren and M. Russkikh)

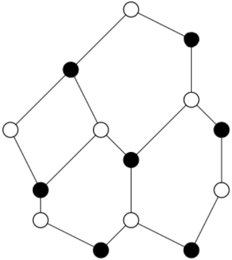
Outline

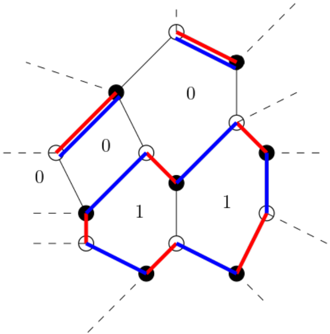
1. Dimer model basics
2. Hexagons, lozenge tilings, and the height function
3. t-embeddings in general
4. Perfect t-embeddings of Hexagons

Dimer model basics

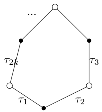
Dimer model

1. We study random perfect matchings M of a bipartite planar weighted graph $(\mathcal{G}, \nu)$. The probability measure is $\mu_{\mathcal{G}}(M) = \frac{1}{Z} \prod_{e \in M} \nu_e$.
2. Define the height function $h = h_M$ by superimposing M with a reference matching M_0 and viewing the loops as level lines of h.
3. **Goal:** Study asymptotic height fluctuations $h - \mathbb{E}[h]$.





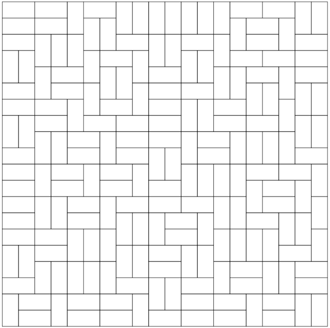
Kasteleyn signs and local statistics

1. **Complex Kasteleyn weighting:** We choose $|\tau_e| = 1, e \in E$ such that around any face
$$\frac{\tau_1}{\tau_2}\frac{\tau_3}{\tau_4}\cdots\frac{\tau_{2k-1}}{\tau_{2k}} = (-1)^{k+1}.$$


2. Using edge weights $\{\nu_e\}$ and signs $\{\tau_e\}$, a **Percus-Kasteleyn matrix** K (whose rows index white vertices and columns index black vertices) is defined by $K(w, b) = \tau_{bw}\nu_{bw}$.
   - **Theorem [Percus '69, Kasteleyn '61]:** $Z = |\det K|$.
   - **Corollary [Kenyon]:** Let $E' = \{e_i = (b_i, w_i)\}_{i=1}^k$ be a set of edges. Then
$$\mu_{\mathcal{G}}(\{M : E' \subset M\}) = \left(\prod_{i=1}^k K(w_i, b_i)\right)\det(K^{-1}(b_i, w_j))_{i,j=1}^k.$$

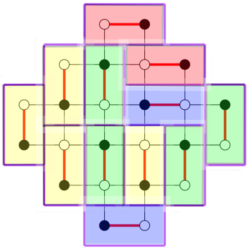
Dimer model on square lattice

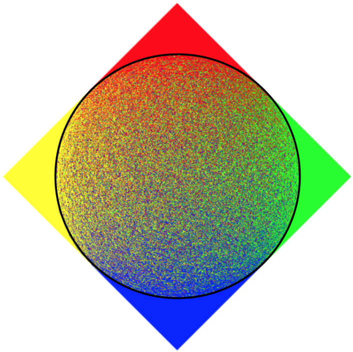
1. Useful point of view: K approximates $\overline{\partial}$.
2. In certain setups, this leads to proofs of convergence of height function fluctuations to a *Gaussian free field* [Kenyon'00, Russkikh'18], a *conformally invariant limit*.



General boundary conditions

1. Useful point of view: K approximates $\overline{\partial}$.
2. In general, can still approximate $K \sim \overline{\partial}$, though this approximation happens in a nontrivial way. E.g. even at leading order we see *phase separation*.
3. Big goal: use *t-embeddings* to do this. (More on this later.)

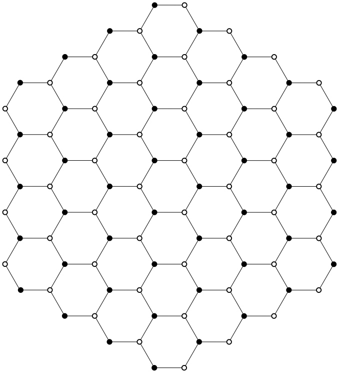


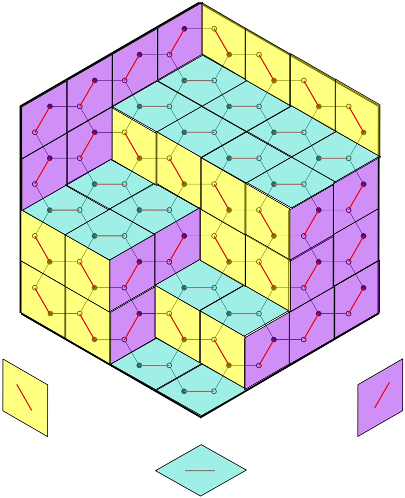


Honeycomb dimer model on the hexagon

Hexagon

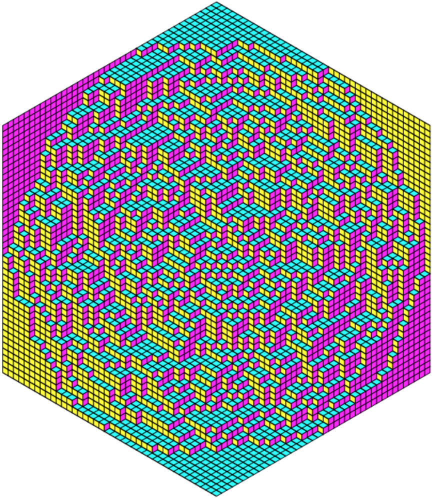
1. We consider the $N \times N \times N$ **Hexagon**: (See below for $N = 4$).
2. Perfect matchings are *lozenge tilings of a hexagon*.





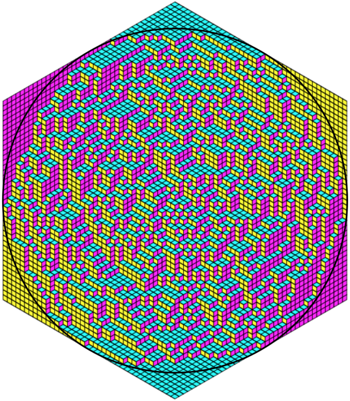
Hexagon

1. The *height function* of a matchings is obtained from viewing the picture as "stacks of cubes".
2. As $N \to \infty$, how does a random tiling behave?



Hexagon Dimer Model: Limit Shape

1. **Theorem [Cohn-Larsen-Propp '97]:** The rescaled height $\frac{1}{N}h_N(Nx, Ny)$ converges in probability to a **deterministic** limit $\mathfrak{h}(x, y)$.
2. The region where the slopes of $\mathfrak{h}$ are strictly between 0 and 1 is called the *rough region*. Outside the rough region, fluctuations of h_N are exponentially suppressed.



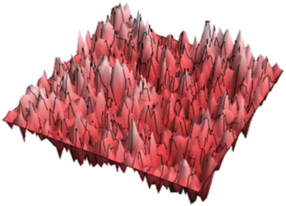
Fluctuations: Gaussian free field

1. The Gaussian free field on a domain $D \subset \mathbb{R}^2$ is the Gaussian process on D with covariance structure given by the Greens function: for any $x_1, \ldots, x_k \in D$, $(F_D(x_i))_{i=1}^k$ is a mean 0 Gaussian vector and

$$\mathbb{E}[F_D(x_i)F_D(x_j)] = G_D(x_i, x_j)$$

where G_D is the Dirichlet Greens function on D.

2. Note G_D is singular on the diagonal; F_D is really a random distribution (not a function).

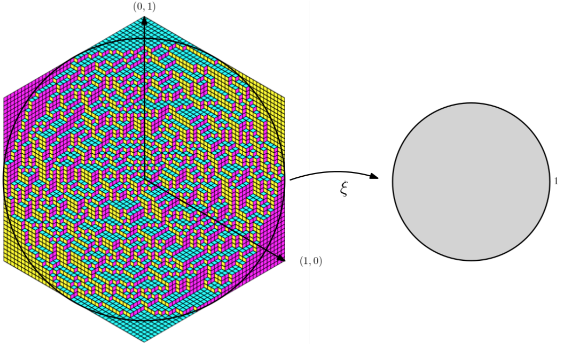


Fluctuations: Gaussian free field

1. Exists a diffeomorphism $\xi : \{(x, y) \in \text{Inscribed disc}\} \to \mathbb{D} \subset \mathbb{C}$, the *uniformizing map*. It is **not a conformal map**.
2. **Theorem [Petrov '12]:** The fluctuations $\tilde{h}_N := h_N - \mathbb{E}[h_N]$ converge to a GFF in the complex structure defined by $\xi(x, y)$:

$$\tilde{h}_N \to F \circ \xi$$

where $F = F_{\mathbb{H}}$ is a Dirichlet GFF on $\mathbb{D}$.

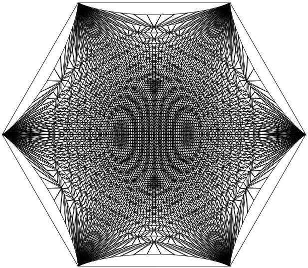


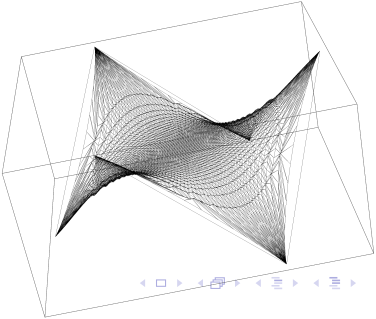
Fluctuations: a general conjecture

1. **A general conjecture of [Kenyon-Okounkov '07] (dimer models with $\mathbb{Z}^2$-periodic weights):** Height fluctuations converge to a GFF, generically in a nontrivial **complex structure** which depends in a nontrivial way on boundary conditions.
2. **Goal of studying new "t-embeddings"** introduced and studied by [Affolter '18], [Kenyon-Lam-Ramassamy-Russkikh '18]: To try to prove convergence to GFF using *discrete holomorphicity*; a theory of discrete holomorphic functions on t-embeddings is developed in [Chelkak-Laslier-Russkikh '20].
3. This is used to **prove convergence to GFF**, assuming existence of "perfect" t-embeddings defined in [Chelkak-Laslier-Russkikh '21]

Fluctuations: a general theorem [Chelkak, Laslier, Russkikh]

1. **t-embeddings: a new graph embedding.** Each t-embedding comes with a discrete surface; the graph of the *origami map* $\mathcal{O}$.
2. If given *perfect* ([CLR '21]) t-embeddings $\mathcal{T}_n$ which satisfy two nondegeneracy assumptions, and if the discrete surfaces $(\mathcal{T}_n, \mathcal{O}_n)$ converge to a **maximal surface in Minkowski space** $S \subset \mathbb{R}^{2,1}$, then height function fluctuations converge to Gaussian Free Field in the conformal structure of S.





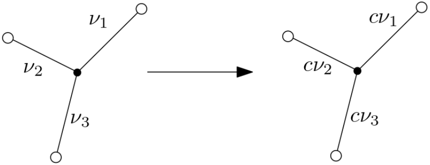
t-embeddings

Gauge equivalence

1. Recall we have a weight function $\nu : E \to \mathbb{R}_{>0}$. The **Boltzmann measure** on dimer configurations is

$$\mu(M) = \frac{1}{Z} \prod_{e \in M} \nu_e.$$

2. Two weight functions are *gauge equivalent* if one can be obtained from the other by a sequence of measure-preserving local operations, as below:

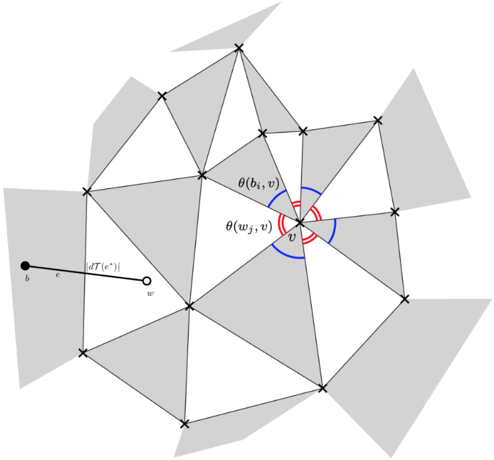


Definition: t-embedding

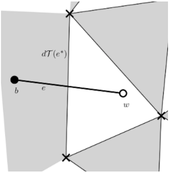
A t-embedding $\mathcal{T} : \mathcal{G}^* \to \mathbb{C}$ of a bipartite graph $(\mathcal{G}, \nu)$ satisfies:

1. **Properness**, **Gauge equivalence** (the edge weights are geometric, and induce the same statistics), **Angle condition**

Figure from M. R.



Geometric Kasteleyn weights



1. A t-embedding defines a Kasteleyn matrix for $\mathcal{G}$: $K_{\mathcal{T}}(w,b) = d\mathcal{T}((wb)^*)$.
2. In fact, if K is a real Kasteleyn matrix of $\mathcal{G}$, the **angle condition** implies existence of *Coulomb gauges* $F(b)$ and $G(w)$ such that

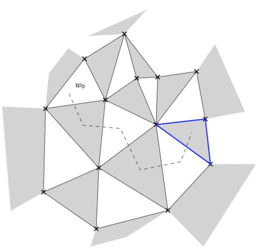
$$d\mathcal{T}((wb)^*) = G(w)K(w,b)F(b)$$

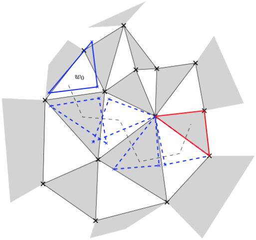
where $\sum_w G(w)K(w,b) = 0$ (and similarly $KF = 0$).
In other words: $K_{\mathcal{T}}$ is a valid Kasteleyn matrix.

$$\mu(\{M : e_i \in M \; \forall i\}) = \left(\prod_{i=1}^{k} K_{\mathcal{T}}(w_i, b_i)\right) \det(K_{\mathcal{T}}^{-1}(b_i, w_j))_{i,j=1}^{k}.$$

Definition: Origami map

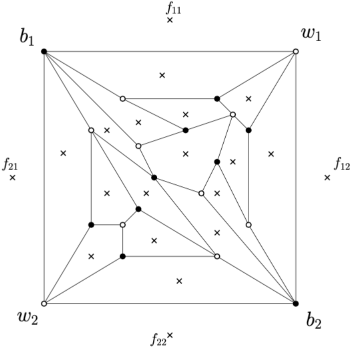
1. The origami map is a piecewise smooth map $Image(\mathcal{T}) \subset \mathbb{C} \to \mathbb{C}$.
2. Geometrically, to compute $\mathcal{O}(b)$ for a face b, fold across edges of $\mathcal{T}$ along a face path to a reference face w_0:





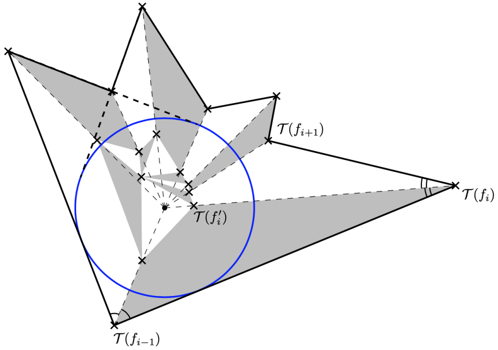
Boundary conditions: Augmented dual graph

1. In fact, we embed the *augmented dual graph*.



Definition: Perfect t-embeddings

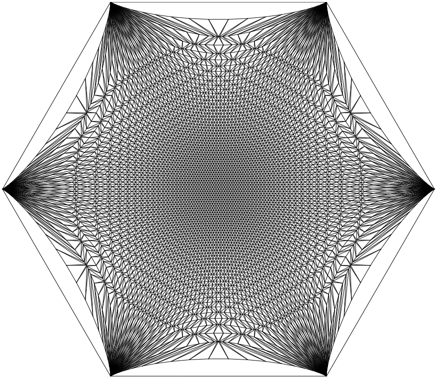
1. Recall we have a boundary polygon P. A t-embedding is **perfect** if
   - P is tangential to the unit disk;
   - The dual edges connecting outer dual vertices to dual vertices in the interior of $\mathcal{G}^*$ are angle bisectors.



Fluctuations: a general theorem [Chelkak, Laslier, Russkikh]

1. Recall: given *perfect* ([CLR '21]) t-embeddings $\mathcal{T}_n$ which satisfy two nondegeneracy assumptions, and that the discrete surfaces $(\mathcal{T}_n, \mathcal{O}_n)$ converge to a **maximal surface in Minkowski space** $S \subset \mathbb{R}^{2,1}$, then height function fluctuations converge to Gaussian Free Field in the conformal structure of S.
2. General existence of perfect t-embeddings is a conjecture, known if outer face is of degree 4.
3. Goal: Understand how to construct and analyze perfect t-embeddings.

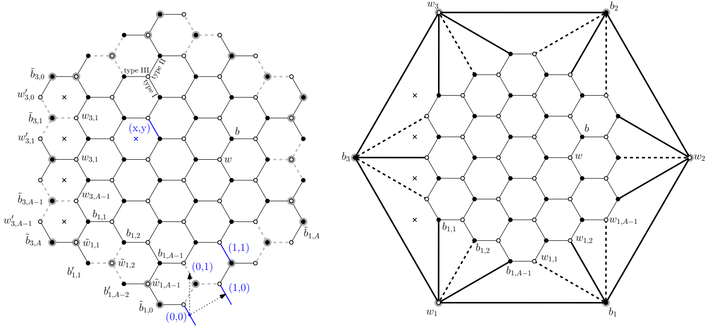
Uniform Hexagon



How to build this t-embedding?

Construction of t-embedding for (reduced) Hexagon

1. We construct an embedding of H'_A, the $A \times A \times A$ *reduced hexagon*.



2. The dimer statistics inside are the same as for uniformly random lozenge tilings.

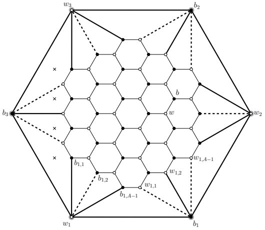
Gauge functions in terms of $K_{\mathbb{R}}^{-1}$

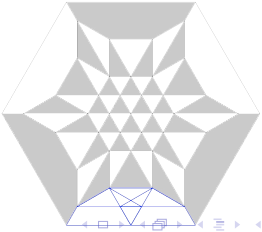
Theorem (Berggren, N., Russkikh)

The Coulomb Gauge functions define a perfect t-embedding of H'_A:

$$G(w) = \sum_{j=1}^{3} e^{-\mathrm{i}j2\pi/3} K_{\mathbb{R}}^{-1}(b_j, w)$$

$$F(b) = -\sum_{j=1}^{3} e^{-\mathrm{i}(j-1)2\pi/3} K_{\mathbb{R}}^{-1}(b_j, w)$$





t-embeddings: a contour integral formula

Using exact formulas of [Pet '12] for $K_{\mathbb{R}}^{-1}$, we obtain

Theorem (Berggren, N., Russkikh)

For all faces (x, n) in the hexagon

$$\mathcal{T}_A(x,n) = \frac{1}{(2\pi i)^2}\int_{\mathcal{E}_2}\int_{\mathcal{E}_1} e^{A(S(w)-S(z))} f(w)g(z)\frac{dz\,dw}{z-w}. \quad (1)$$

Above $f(w), g(z)$ are rational functions, and $S(z) = S(z; \frac{x}{A}, \frac{n}{A})$, where $S(z; \chi, \eta)$ is the same action function appearing in [Petrov '12]. A similar formula for $\mathcal{O}$ is obtained.

1. This expression is amenable to asymptotic analysis.
2. As shown in [Pet '12], the action function is related to the conformal structure of the limiting Gaussian free field. The *critical point* $\zeta(\chi, \eta)$ is the uniformizing map.

Asymptotic analysis of t-embeddings: maximal surface

Theorem (Berggren, N., Russkikh)

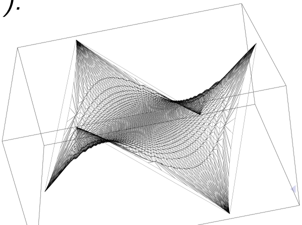
1. *We have*

$$\lim \mathcal{T}_A(A\chi, A\eta) = z(\chi,\eta) = \frac{1}{2\pi \mathrm{i}} \int_{\bar{\zeta}}^{\zeta} f(z) g(z) dz$$

$$\lim \mathcal{O}_A(A\chi, A\eta) = \vartheta(\chi,\eta) = \frac{1}{2\pi \mathrm{i}} \int_{\bar{\zeta}}^{\zeta} f(z) \overline{g(\bar{z})} dz.$$

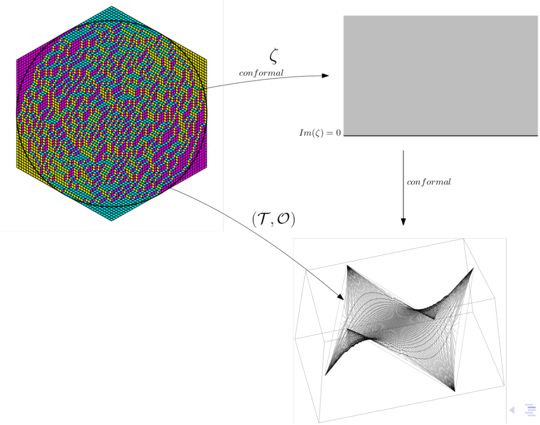
where f, g are explicit and $\zeta = \zeta(\chi,\eta)$ is the critical point.

2. *The limiting graph $\{z, \vartheta(z)\}$ is a maximal surface in $\mathbb{R}^{2,1}$ (has mean-curvature zero with respect to the ambient metric $|dz|^2 - d\vartheta^2$).*



Asymptotic fluctuations via t-embeddings

1. Our results + [CLR '21, Theorem 1.4] imply (a new proof of): height function fluctuations converge to Gaussian Free Field in the conformal structure of the surface. Is it consistent with [Pet '12]?
2. Our formula for $(\mathcal{T}_A(A\chi, A\eta), \mathcal{O}_A(A\chi, A\eta))$ is a composition:



Open questions

1. Existence and uniqueness! (See [CLR '21] for details and precise conjectures.)
2. Can we find existence via a variational principle? C.f. works of Thurston, Bobenko, Rivin, Colin de Verdiere and others on circle packings, circle patterns, discrete holomorphic maps, etc...
3. What is the mechanism for the appearance of a maximal surface?

End!