Positivity everywhere

Lecture 1

Natasha Blitvić

(Queen Mary University of London)

TA: Slim Kammoun (ENS $\mapsto$ Poitiers)

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Rough format:

Lectures — bird's eye view

Tutorials — work out the details

Many interconnected themes: choose your own adventure!

Good to keep in mind:

- look to understand (familiar) ideas from multiple perspectives
- look for interesting juxtapositions

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Part I : Positivity (non-negativity) everywhere

Algebraic/combinatorial object $\rightsquigarrow$ Representing matrix (?) $\rightsquigarrow$ Positivity of the matrix (?)

$[a_{ij}]$ s.t. $a_{ij} \geq 0 \quad \forall\, i,j$ (Maybe $k \times m$, maybe infinite)

$A = [a_{ij}] = [\overline{a_{ji}}]$ s.t. $\langle Av, v \rangle \geq 0 \quad \forall\, v$

$\det[a_{ij}]_{i,j \in I} \geq 0 \quad \forall\, I$ Principal minors

$\det[a_{ij}]_{1 \leq i,j \leq n} \geq 0 \quad \forall\, n$ Principal leading minors

$\det[a_{ij}]_{i \in I, j \in J} \geq 0 \quad \forall\, |I| = |J|$ All minors

$\det[a_{ij}]_{i \in I, j \in J} \geq 0 \quad \forall\, |I| = |J| = 2$ 2×2 minors

$\det[a_{ij}]_{i \in I, j \in J} \geq 0 \quad \forall\, |I| = |J| = \text{rank}$ Maximal minors

Notation: $[n] := \{1, 2, \ldots, n\}$

Def A matrix is **totally positive** if all of its minors are non-negative.

Example 1

Birth-death processes

$$\begin{tikzcd} 0 \arrow[r, bend left, "\lambda_0"] & 1 \arrow[l, bend left, "\mu_1"] \arrow[r, bend left, "\lambda_1"] & 2 \arrow[l, bend left, "\mu_2"] \arrow[r, bend left, "\lambda_2"] & 3 \arrow[l, bend left, "\mu_3"] & \cdots \end{tikzcd}$$

$$p_{j,k}(t) = \mathbb{P}(X(t) = k \mid X(0) = j)$$

$$p_{k,k+1}(\Delta t) = \lambda_k \Delta t + o(\Delta t)$$

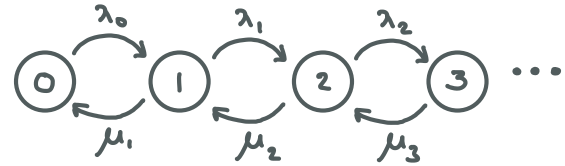
$$p_{k,k-1}(\Delta t) = \mu_k \Delta t + o(\Delta t)$$

$$p_{k,k}(\Delta t) = 1 - (\lambda_k + \mu_k)\Delta t + o(\Delta t)$$

Thm (Karlin & McGregor '57) For any $t > 0$,

$[p_{j,k}(t)]_{j,k\geq 0}$ is totally positive.

Proof sketch #1 (KM'57, §5):



$$\left.\begin{aligned}
P_{k,k+1}(\Delta t) &= \lambda_k \Delta t + o(\Delta t)\\
P_{k,k-1}(\Delta t) &= \mu_k \Delta t + o(\Delta t)\\
P_{k,k}(\Delta t) &= 1 - (\lambda_k + \mu_k)\Delta t + o(\Delta t)
\end{aligned}\right\}
\qquad
\begin{aligned}
P_n(t) &= \left[p_{j,k}(t)\right]_{0 \le i,j \le n}\\
P_n(0) &= I_n\\
\frac{d}{dt} P_n(t) &= A_n P_n(t)
\end{aligned}$$

where
$$A_n = \begin{bmatrix}
-(\lambda_0+\mu_0) & \lambda_0 & & & 0\\
\mu_1 & -(\lambda_1+\mu_1) & \ddots & & \\
 & \mu_2 & \ddots & \ddots & \\
 & & \ddots & \ddots & \lambda_{n-1}\\
0 & & & \mu_n & -(\lambda_n+\mu_n)
\end{bmatrix}$$

Observe $I + \frac{t A_n}{n}$ is TP for n large enough.

Deduce e^{tA_n} is TP $\forall\, t>0$.

Some further questions you could ask:

- Which solutions P satisfying the backward equation

  $\frac{d}{dt} P(t) = A P(t)$ and the fwd eq $\frac{d}{dt} P(t) = P(t) A$

  $P(0) = I$ are transition matrices of some Markov process?

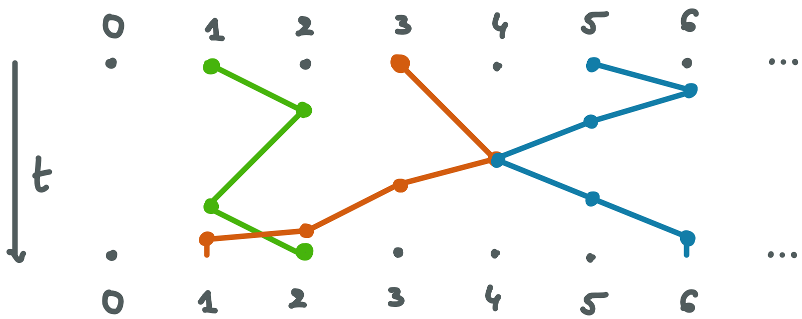
- If we start with a time-differentiable matrix of

  probabilities and if the matrix has "enough positivity",

  will its time derivative be of the tridiagonal form?

(See KM '57)

Proof sketch #2 (KM'59): n particles executing the birth death process



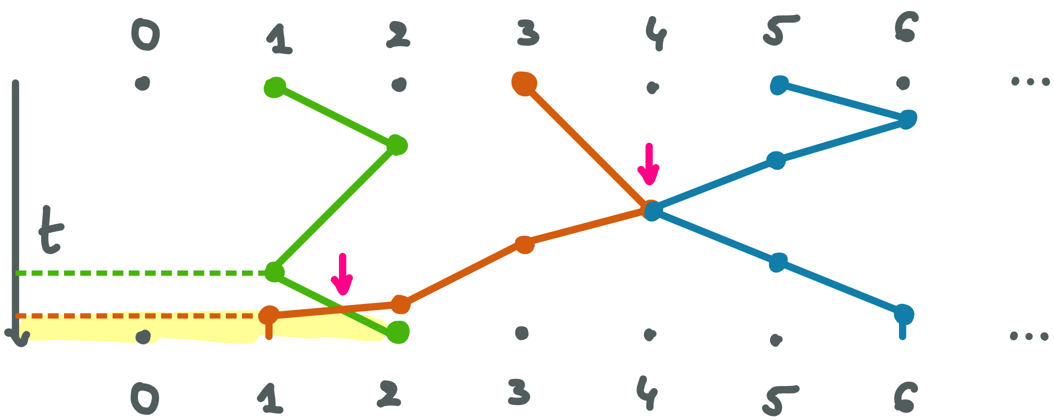
$$\det\left[p_{ij}(t)\ ;\ \begin{matrix} i_1 < i_2 < \cdots < i_n \\ j_1 < j_2 < \cdots < j_n \end{matrix}\right] = \sum_{\sigma \in S_n} (-1)^{\mathrm{inv}(\sigma)} \, p_{i_1, j_{\sigma(1)}}(t)\, p_{i_2, j_{\sigma(2)}}(t) \cdots p_{i_n j_{\sigma(n)}}(t)$$

Claim:

$$\det\left[p_{ij}(t) .\ \begin{matrix} i_1 < i_2 < \cdots < i_n \\ , j_1 < j_2 < \cdots < j_n \end{matrix}\right] = \text{Prob}\left(\text{at time } t, \text{ particles found in } j_1, j_2, \dots, j_n \text{ without having coincided in any state}\right)$$

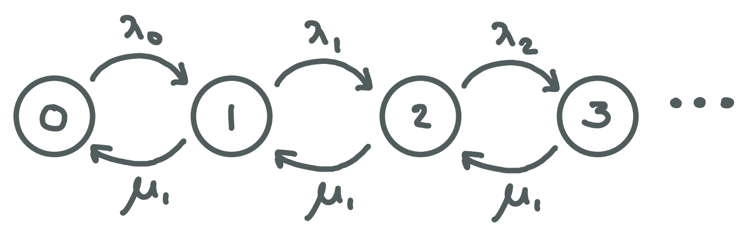
$$\det\left[p_{ij}(t) \; : \; \begin{matrix} i_1 < i_2 < \dots < i_n \\ j_1 < j_2 < \dots < j_n \end{matrix}\right] = \sum_{\sigma \in S_n} (-1)^{\mathrm{inv}(\sigma)} \, p_{i_1 j_{\sigma(1)}}(t) \cdots p_{i_n j_{\sigma(n)}}(t) =$$

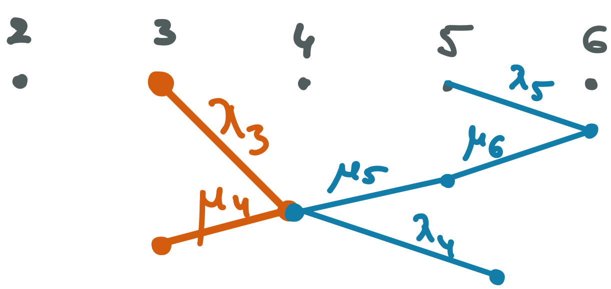
prob (at time t, particles found in $j_1, j_2, \dots, j_n$ resp. without having coincided in any state)



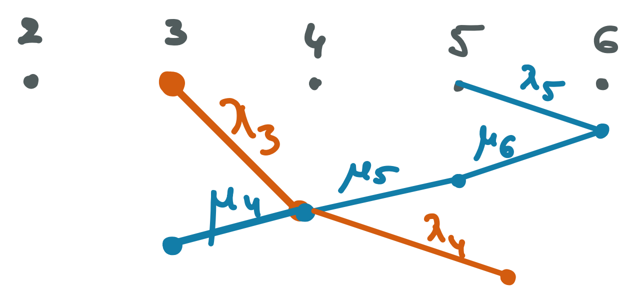
Proof by example:

$p_{ij}(t)$ computed from





vs



same overall probability but opposite signs

Some further questions you could ask:

- Where did we use the fact that the process is birth-death?

- What about a general Markov process on $\mathbb{N}_0$.

  Exercise: Write down formula for the determinant

- Did we need to have probabilities, or even positive weights?

Independently: Gessel-Viennot '85 based on Lindström '73

$G = (V, E)$ locally finite edge-weighted directed acyclic graph,

$A = \{a_1, \dots, a_n\}$, $B = \{b_1, \dots, b_n\} \subseteq V$ (need not be disjoint)

weight of a path p = $wt(p)$ = product of edge weights

$$W = [w_{ij}]_{1 \le i,j \le n} \quad \text{where } w_{ij} = \sum_{\substack{\text{paths } p \\ a_i \mapsto b_j}} wt(p)$$

<u>Lemma (LGV)</u>

$$\det W = \sum_{\sigma \in S_n} (-1)^{inv(\sigma)} \sum_{\substack{\text{vertex-disjoint paths} \\ p_1 : a_1 \mapsto b_{\sigma(1)} \\ \vdots \\ p_n : a_n \mapsto b_{\sigma(n)}}} wt(p_1) \cdots wt(p_n)$$

Proof:

same idea (exercise)

Example 2

Matroids: unify several notions of independence

Def Matroid $M = (E, \mathcal{B})$, where E is a finite set ("ground set")

and $\mathcal{B} \subseteq 2^E$ ("bases of M"), $\mathcal{B} \neq \emptyset$, s.t. $\forall B_1, B_2 \in \mathcal{B}$

and $b_1 \in B_1 - B_2$, $\exists\, b_2 \in B_2 - B_1$ w/ $(B_1 - \{b_1\}) \cup \{b_2\} \in \mathcal{B}$.

("basis exchange axiom")

E.g. $A \in \mathrm{Mat}_{d \times n}(K)$, $\mathrm{rank}(A) = d$, $A = (a_1, a_2, \dots, a_n)$.

Let $\mathcal{B} = \{ B \subseteq [n] \mid \{a_i\}_{i \in B}$ form a linear basis for $K^d \}$.

Check: $M(A) = ([n], \mathcal{B})$ is a matroid.

Such a matroid is representable.

Def. (Postnikov) Positroid : a matroid on [n] representable by columns of a real matrix, whose maximal minors are non-negative.

Positroids $\longleftrightarrow$ Decorated permutations e.g. 1536427

$\longleftrightarrow$ Grassmann necklaces

$\longleftrightarrow$ Ꞁ-diagrams

$\longleftrightarrow$ equiv. class. of plabic graphs

(Postnikov '06
Oh '11)

Many matroidal properties: closure properties, duality

Example 3 (subtler)

From now on: $(a_n)_{n\geq 0}$ denotes a real sequence

Def $(a_n)_n$ is **unimodal** if $a_0 \leq a_1 \leq \dots \leq a_k \geq a_{k+1} \geq \dots$
for some $0 \leq k \leq n$.

Def (a_n) is **log-concave** if $a_k^2 \geq a_{k-1} a_{k+1} \quad \forall k$

Def (a_n) is **log-convex** if $a_k^2 \leq a_{k-1} a_{k+1} \quad \forall k$

Example: $\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \dots, \binom{n}{n-1}, \binom{n}{n}$

- unimodal
- log-concave: $$\frac{\binom{n}{k}^2}{\binom{n}{k-1}\binom{n}{k+1}} = \frac{(n-k+1)(k+1)}{(n-k)k} > 1$$

Exercise: if $a_n > 0 \ \forall n$, log-concavity $\Rightarrow$ unimodality.

Notation: $[n]_q := \dfrac{1-q^n}{1-q} = 1+q+\cdots+q^{n-1}$ with $[0]_q := 0$

$[n]_q! := [n]_q [n-1]_q \cdots [2]_q [1]_q$ with $[0]_q! := 1$

Exercise: Show that

$$\begin{bmatrix} n \\ 0 \end{bmatrix}_q, \begin{bmatrix} n \\ 1 \end{bmatrix}_q, \ldots, \begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[n-k]_q!\,[k]_q!}, \ldots, \begin{bmatrix} n \\ n \end{bmatrix}_q$$

is log-concave for $q \geq 0$.

Thm (Huh '09) Consider a matroid M representable over a field of characteristic 0 with characteristic polynomial

$$\chi_M(q) = \mu_0 q^{r+1} - \mu_1 q^r + \cdots + (-1)^{r+1} \mu_{r+1},$$

The sequence $\mu_0, \ldots, \mu_{r+1}$ is log-concave.

Proves a conjecture of Read ('68) that chromatic polynomials of graphs are unimodal. More generally:

(ex.) **Conj** (Rota, Heron, Walsh ~'70)

Thm Adiprasito, Huh, Katz '15

Coefficients of the characteristic polynomial of any finite matroid form a log-concave sequence.

Toeplitz and Hankel matrices

<u>Def</u> To a real seq. $(a_n)_{n\geq 0}$, we can associate the infinite

Toeplitz matrix and Hankel matrix.

$$\begin{pmatrix} a_0 & & & & 0 \\ a_1 & a_0 & & & \\ a_2 & a_1 & a_0 & & \\ a_3 & a_2 & a_1 & a_0 & \\ \vdots & \vdots & & \ddots & \ddots \\ a_d & a_{d-1} & \cdots & a_1 & a_0 \\ \vdots & \vdots & & & \ddots \end{pmatrix} \qquad \begin{pmatrix} a_0 & a_1 & a_2 & a_3 & \cdots \\ a_1 & a_2 & a_3 & a_4 & \cdots \\ a_2 & a_3 & a_4 & a_5 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \\ a_d & a_{d-1} & \cdots & & \\ \vdots & \vdots & & & \end{pmatrix}$$

$$T(a) = \begin{pmatrix} 1 & & & & & \\ a_1 & 1 & & & 0 & \\ a_2 & a_1 & 1 & & & \\ a_3 & a_2 & a_1 & 1 & & \\ \vdots & \vdots & & \ddots & \ddots & \\ a_d & a_{d-1} & \cdots & & a_1 & 1 \\ \vdots & \vdots & & & & \ddots \end{pmatrix}$$

<u>Thm</u> (Schoenberg, Aissen-Schoenberg-Whitney, Edrei '48–'53
See also Thoma '64)

$T(a)$ is totally positive iff in some nbhd of $z=0$

$$1 + a_1 z + a_2 z^2 + \cdots = e^{z\alpha} \prod_{i\geq 0} \frac{(1+\beta_i z)}{(1-\gamma_i z)}$$

for $\alpha \geq 0$, $\beta_1 \geq \beta_2 \geq \ldots \geq 0$, $\gamma_1 \geq \gamma_2 \geq \ldots \geq 0$

with $\sum_i \beta_i + \sum_i \gamma_i < \infty$.

Rietsch '01: parametrization of $n\times n$ totally positive Toeplitz

$$T(a) = \begin{pmatrix} a_0 & & & & \\ a_1 & a_0 & & 0 & \\ a_2 & a_1 & a_0 & & \\ a_3 & a_2 & a_1 & a_0 & \\ \vdots & \vdots & & \ddots & \ddots \\ a_d & a_{d-1} & \cdots & & \\ \vdots & \vdots & & & \end{pmatrix}$$

Observe:

$T(a)$ totally positive

$\Rightarrow$ $a_0, a_1, \cdots$ is log-concave

E.g. $\binom{n}{0}, \cdots, \binom{n}{k}, \cdots, \binom{n}{n}$

$\left\langle {n \atop 0} \right\rangle, \cdots, \left\langle {n \atop k} \right\rangle, \cdots, \left\langle {n \atop n} \right\rangle$ Eulerian #'s, e.g. $\sigma \in S_n$ w/ k ascents

$\left[{n \atop 0} \right], \cdots, \left[{n \atop k} \right], \cdots, \left[{n \atop n} \right]$ Stirling #1, e.g. $\sigma \in S_n$ w/ k cycles

$\left\{ {n \atop 0} \right\}, \cdots, \left\{ {n \atop k} \right\}, \cdots, \left\{ {n \atop n} \right\}$ Stirling #2, e.g. partitions of $[n]$ into k parts

What about log convexity?

Some log-convex sequences: $a_k^2 \leq a_{k-1} a_{k+1}$

$n!$ e.g. permutations

B_n Bell #'s e.g. set partitions

C_n Catalan #'s e.g. non-crossing set partitions

But also:

$$\left(\sum_{k=0}^{n} \binom{n}{k} x^k \right)_{n \geq 0} \quad \forall x \in \mathbb{R} \quad \text{(trivial)}$$

Eulerian $$\left(\sum_{k=0}^{n} \left\langle {n \atop k} \right\rangle x^k \right)_{n \geq 0} \quad (x \geq 0)$$

Stirling II $$\left(\sum_{k=0}^{n} \left\{ {n \atop k} \right\} x^k \right)_{n \geq 0} \quad (x \geq 0)$$

Def A real sequence $(a_n)_{n\geq 0}$ is

- **Toeplitz totally positive** if $T(a) = \begin{pmatrix} a_0 & & & & & 0 \\ a_1 & a_0 & & & & \\ a_2 & a_1 & a_0 & & & \\ a_3 & a_2 & a_1 & a_0 & & \\ \vdots & \vdots & & & \ddots & \\ a_d & a_{d-1} & \cdots & & a_1 & a_0 \\ \vdots & \vdots & & & & \ddots \end{pmatrix}$ is totally positive.

- **Hankel totally positive** if $H(a) = \begin{pmatrix} a_0 & a_1 & a_2 & a_3 & \cdots \\ a_1 & a_2 & a_3 & a_4 & \cdots \\ a_2 & a_3 & a_4 & a_5 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \\ a_d & a_{d-1} & \cdots & & \\ \vdots & \vdots & & & \end{pmatrix}$ is totally positive.

Exercise:

Toeplitz TP $\Rightarrow$ log-concave

Hankel TP $\Rightarrow$ log-convex

Take your favorite combinatorial sequence (a_n).

(From now on, $a_0 = 1$)

When does there exist a probability measure μ on $\mathbb{R}$ s.t.

$$a_n = \int_{\mathbb{R}} x^n \, d\mu(x) \quad \forall n \in \mathbb{N} \; ?$$

Thm (Hamburger 1920-21)

$(a_n)_{n\geq 0}$ is a moment sequence of a positive Borel measure μ on $\mathbb{R}$ iff $\forall k \in \mathbb{N}$, $\forall z_0, z_1, \dots, z_k \in \mathbb{C}$,

$$\sum_{j,l=0}^{k} a_{j+l} z_j \cdot \overline{z_l} \geq 0 ,$$

i.e. the Hankel matrices $[a_{i+j}]_{i,j\leq n}$ are positive semidefinite $\forall n$.

<proof>

$(\Rightarrow)$

$$\sum_{j,l=0}^{k} a_{j+l} z_j \cdot \overline{z_l} = \int_{\mathbb{R}} \underbrace{\left| \sum_{j=0}^{k} z_j \cdot x^j \right|^2}_{\geq 0} \underbrace{d\mu(x)}_{\geq 0}$$

$(\Leftarrow)$ Subsequent lecture

<u>Thm</u> (Stieltjes 1894-95, Gantmakher-Krein '1937) TFAE:

(1) $\exists\ \mu \geq 0$ on $[0,\infty)$ s.t. $a_n = \int_{[0,\infty)} x^n \, d\mu(x)$

(2) The infinite Hankel matrix $\begin{bmatrix} a_0 & a_1 & a_2 & \cdots \\ a_1 & a_2 & a_3 & \cdots \\ a_2 & a_3 & \cdots & \\ \vdots & & & \end{bmatrix}$ is totally positive.

Recap: $a_0 = 1$

$$H_n(a) = \begin{bmatrix} a_0 & a_1 & \cdots & a_n \\ a_1 & a_2 & \cdots & a_{n+1} \\ \vdots & & & \\ a_n & a_{n+1} & \cdots & a_{2n} \end{bmatrix}$$

$H_n(a)$ positive semidefinite $\forall n$ $\iff$ a_n is a sequence of moments of a probability measure on the real line

(Hamburger moment problem)

$H(a)$ totally positive $\iff$ a_n is a sequence of moments of a probability measure on $[0,\infty)$

(Stieltjes moment problem)

$\implies$ a_n is log-convex

$T(a)$ totally positive $\implies$ a_n is log-concave

Positivity is natural:

$$n! \quad , \quad C_n \quad , \quad B_n$$

$$\binom{n}{0}, \dots, \binom{n}{k}, \dots, \binom{n}{n}$$

$$\left\langle {n \atop 0} \right\rangle, \dots, \left\langle {n \atop k} \right\rangle, \dots, \left\langle {n \atop n} \right\rangle$$

$$\left[{n \atop 0} \right], \dots, \left[{n \atop k} \right], \dots, \left[{n \atop n} \right]$$

$$\left\{ {n \atop 0} \right\}, \dots, \left\{ {n \atop 0} \right\}, \dots, \left\{ {n \atop 0} \right\}$$

$$\vdots$$

But not to be expected:

- # matroids on $[n]$: $1, 2, 4, 8, 17, 38, 98, \ldots$ A055545
- # binary matroids on $[n]$: $1, 2, 4, 8, 16, 32, 68, 148, \ldots$ A076766
- # ternary matroids on $[n]$: $2, 4, 8, 17, 36, 85, \ldots$ A076892
- # simple matroids on $[n]$: $1, 2, 4, 9, 26, \ldots$ A002773

$\vdots$

and many more examples (coming soon) that are **NOT** moment sequences.

Positivity shows up in unexpected places:

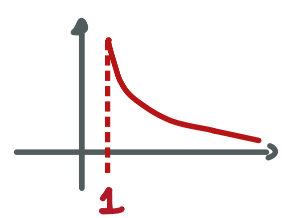
<u>Thm</u>

\# positroids on $[n]$

$=$ \# decorated permutations on $[n]$ (Postnikov)

$=$ n^{th} moment of $1 + \mathrm{Exp}(1)$ (Ardila, Rincón, Williams '16)

$$\left(= \int_1^\infty x^n e^{-(x-1)}\, dx \right)$$



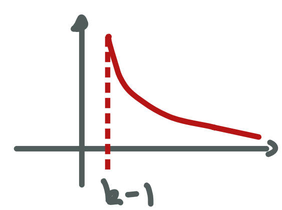
Def (B. - Steingrímsson '21) A k-arrangement on $[n]$ is a permutation $\sigma \in S_n$, together with a k-coloring of its fixed points.

Remark: $k=2$ $\equiv$ decorated permutations

Thm (B. - Steingrímsson '21)

$$\#\, k\text{-arrangements on } [n] = n^{th} \text{ moment of } k-1+\mathrm{Exp}(1)$$

$$\left(= \int_1^\infty x^n e^{-(x-k+1)}\, dx \right)$$



<proof> Generating functions.

$\rightsquigarrow$ Various other combinatorial properties

$\rightsquigarrow$ Further unexpected occurence in a different probabilistic setting

} Lec 4

Recall:

$[p_{j,k}(t)]_{j,k\geq 0}$ is totally positive $\forall\, t>0$

See Karlin & McGregor '57 for consequences

'59 for generalizations

Lec 4 will contain recent examples of a probabilistic "artifact" carrying additional probabilistic structure.

Positivity shows up in unexpected places:

Observation / Program of work

(B. & Steingrímsson, Elvey Price & Guttmann):

Hard combinatorial problems often display some form of positivity. (Focus: moment sequences)

- Deeper structural understanding
- Better asymptotics
- New tools

Observation / Program of work

(B. & Steingrímsson, Sokal & Zeng, ...):

Moment sequences in combinatorics tend to be "related", e.g. following from some general combinatorial principle

- Unifying frameworks
- Interesting juxtapositions
- New tools / New definitions