Positivity everywhere

Lecture 2

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Themes (how we'll choose to interpret them)

Universality : same prob. structures arising in different contexts.

Integrability : · Explicit formulas
· Algebraic/combinatorial underpinning

Positivity (non-negativity) : When can a set of algebraic/combinatorial objects underpin probabilistic objects?

formal power series

Will enable us to discuss :

- symmetric polynomials / functions
- generating functions
- continued fractions

Def

$R[[x]]$ denotes the algebra of formal power series over ring R

$\sum_{k \in \mathbb{N}_0} a_k x^k$ with $a_0, a_1, \ldots \in R$ equipped with addition:

$$\sum_k a_k x^k + \sum_k b_k x^k = \sum_k (a_k + b_k) x^k$$

and multiplication $\left(\sum_k a_k x^k\right)\left(\sum_k b_k x^k\right) = \sum_k c_k x^k$

where $c_k = \sum_{l=0}^{k} a_l b_{k-l}$.

E.g./Non-example in $\mathbb{C}[[x]]$? $\quad 1 + 2x + \sqrt{2}x^7$,

$$\sum_{k=0}^{\infty} x^k, \qquad \sum_{k=0}^{\infty} k!\, x^k, \qquad \sum_{k=0}^{\infty} (1 + x + x^2 + \cdots + x^k)$$

Equivalently: $R[[x]]$ contains all functions $c: \mathbb{N}_0 \to R$

interpreted as $\sum_{k \in \mathbb{N}_0} c(k)\, x^k$

with addition and multiplication as before.

Multivariate version (in countably many indeterminates):

$R[[x_1, x_2, \ldots]]$ contains all functions $c: \mathbb{N}_0^\infty \to R$ s.t.

$\underline{c(k_1, k_2, \ldots) \neq 0 \Rightarrow k_n = 0 \text{ for all } n \text{ large enough}}$, interpreted as

$$\sum_{\substack{(k_1, k_2, \ldots) \\ \in \mathbb{N}_0^\infty}} c(k_1, k_2, \ldots) \underbrace{x_1^{k_1} x_2^{k_2} \cdots}_{\text{finite degree monomials}}$$

with addition and multiplication defined analogously to multivariate polynomials.

E.g./Non-example? $\sum_{k=1}^{\infty} x_1^k$, $\sum_{k=1}^{\infty} x_k$, $\sum_{k=1}^{\infty} (x_k + x_{k+1})^k$

$(Id, 0, 0, 0, \dots)$ $\mathbb{1}_{\{(1,0,0,\dots),(0,1,0,\dots),(0,0,1,0,\dots),\dots\}}$

$$\sum_{n} \sum_{(k_1,\dots,k_n)} x_{k_1} \cdots x_{k_n}, \quad (1+x_1)(1+x_2)(1+x_3)\cdots$$

Addition, e.g. $\sum_{k=1}^{\infty} x_1^k + \sum_{k=1}^{\infty} k!\, x_k = \sum_{k=1}^{\infty} (x_1^k + k!\, x_k)$

Multiplication, e.g. $\left(\sum_{k=1}^{\infty} x_1^k\right)\left(\sum_{k=1}^{\infty} k!\, x_k\right) = \sum_{k,m=1}^{\infty} m!\, x_1^k x_m$

Differentiation, e.g. $\frac{\partial}{\partial x_1} \sum_{k=1}^{\infty} x_1^k x_2 = \sum_{k=1}^{\infty} k\, x_1^{k-1} x_2$

Sometimes inverses, e.g. $\left(\sum_{k=0}^{\infty} x^k\right)(1-x) = 1$

More on inverses (will be needed later, in univariate setting)

Fact (exercise): $K[[x]]$ where K is a field.

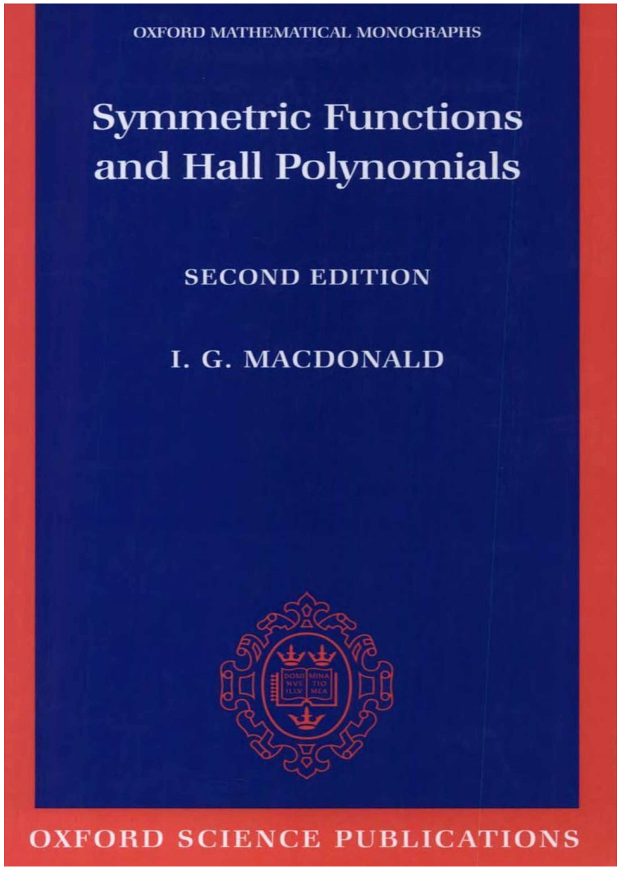
For any fps $\sum_{k=0}^{\infty} a_k x^k$ with $a_0 \neq 0$, $\exists!$ fps $\sum_{k=0}^{\infty} b_k x^k$

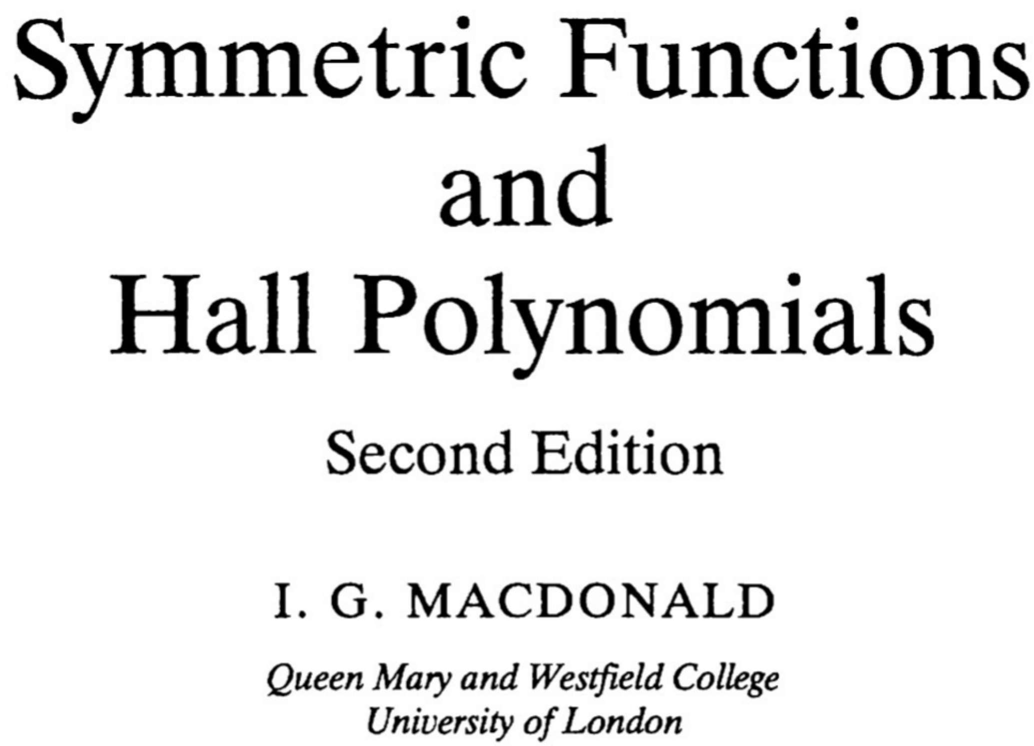
s.t. $\left(\sum_{k=0}^{\infty} a_k x^k\right)\left(\sum_{k=0}^{\infty} b_k x^k\right) = 1.$

In fact,

$$b_0 = a_0^{-1}, \quad b_k = \frac{(-1)^k}{a_0^{k+1}} \det \begin{bmatrix} a_1 & a_0 & & & & \\ a_2 & a_1 & a_0 & & & \\ a_3 & a_2 & a_1 & a_0 & & \\ \vdots & & & & \ddots & a_0 \\ a_k & a_{k-1} & a_{k-2} & a_{k-3} & \cdots & a_1 \end{bmatrix}$$

Positivity everywhere (continued)





Symmetric Functions and Hall Polynomials

Second Edition

I. G. MACDONALD

Queen Mary and Westfield College
University of London

and their many occurences/uses in integrable probability

Degree of a monomial: $x_{k_1}^{d_1} x_{k_2}^{d_2} \cdots x_{k_n}^{d_n} \mapsto d_1 + d_2 + \cdots + d_n$

Projection maps: $\pi_n : R[[x_1, x_2, \dots]] \to R[[x_1, x_2, \dots]]$

$$c(k_1, k_2, \dots) \overset{\pi_n}{\mapsto} c(k_1, k_2, \dots) \, \mathbb{1}_{\{0 = k_{n+1} = k_{n+2} = \dots\}}$$

$$p_d : R[[x_1, x_2, \dots]] \to R[[x_1, x_2, \dots]]$$

$$c(k_1, k_2, \dots) \overset{p_d}{\mapsto} c(k_1, k_2, \dots) \, \mathbb{1}_{\{k_1 + k_2 + \dots = d\}}$$

Together allow us to recover polynomials in $x_1, x_2, \dots, x_n$.

An element $c \in R[[x_1, x_2, \dots]]$ is **homogeneous of degree d** if $p_d(c) = c$, i.e. if c is of the form

$$\sum_{\substack{k_1, k_2, \dots \in \mathbb{N}_0 \\ k_1 + k_2 + \dots = d}} c(k_1, k_2, \dots) \underbrace{x_1^{k_1} x_2^{k_2} \cdots}_{\text{total degree} = d}$$

S_n : group of bijections $[n] \to [n]$ $\quad ([n] := \{1, 2, \dots, n\})$

Write $\sigma \in S_n$ in "one-line notation" : $\sigma(1)\sigma(2)\dots\sigma(n)$

S_∞ : group of bijections $\mathbb{N} \to \mathbb{N}$ of the form

$\sigma(1)\sigma(2)\dots\sigma(n)(n+1)(n+2)(n+3)\dots \quad (n \in \mathbb{N})$

Action of S_∞ on $R[[x_1, x_2, \dots]]$:

For $\sigma \in S_\infty$, $f \in R[[x_1, x_2, \dots]]$:

$$\sigma f = \sigma \sum_{k_1, k_2, \dots \in \mathbb{N}_0} f(k_1, k_2, \dots)\, x_1^{k_1} x_2^{k_2} \cdots$$

$$= \sum_{k_1, k_2, \dots \in \mathbb{N}_0} f(k_1, k_2, \dots)\, x_{\sigma(1)}^{k_1} x_{\sigma(2)}^{k_2} \cdots$$

When $\sigma f = f \ \forall \sigma \in S_\infty$, f is a **symmetric function**.

Check (exercise): $\sigma, \pi \in S_\infty$, $\alpha \in \mathbb{R}$

$$\sigma(f+g) = \sigma f + \sigma g$$

$$\sigma(\alpha f) = \alpha \sigma f$$

$$\sigma(f \cdot g) = (\sigma f) \cdot (\sigma g)$$

$$(\sigma \pi) f = \sigma(\pi f)$$

let Λ = symmetric elements of $\mathbb{R}[[x_1, x_2, \dots]]$

Check (exercise): Λ is a subalgebra of $\mathbb{R}[[x_1, x_2, \dots]]$

From now on, work with symmetric functions over $\mathbb{C}$.

Special families in Λ:

Elementary sym func: $e_n = \sum_{k_1 < k_2 < \cdots < k_n} x_{k_1} x_{k_2} \cdots x_{k_n}$

Complete
Homogeneous sym func: $h_n = \sum_{k_1 \leq k_2 \leq \cdots \leq k_n} x_{k_1} x_{k_2} \cdots x_{k_n}$

Power sum sym func: $p_n = \sum_{k} x_k^n$

Fact (exercise / see Macdonald):

Any one of these families generates Λ

i.e. $\Lambda = \mathrm{alg}\{e_n : n \in \mathbb{N}\} = \mathrm{alg}\{h_n : n \in \mathbb{N}\}$

$= \mathrm{alg}\{p_n : n \in \mathbb{N}\}$

Generating functions:

$$E_n(z) = \sum_{n \geq 0} e_n z^n = \prod_{i \geq 1} (1 + x_i z)$$

$$H_n(z) = \sum_{n \geq 0} h_n z^n = \prod_{i \geq 1} \frac{1}{1 - x_i z}$$

$$P_n(z) = \sum_{n \geq 0} p_{n+1} z^n = \frac{d}{dz} \log \prod_{i \geq 1} \frac{1}{1 - x_i z}$$

(Exercise)

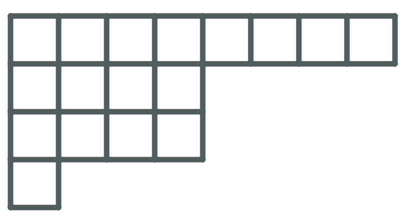
Toward another 'interesting' basis:

A **Young diagram (Ferrers tableau)** $\lambda = (\lambda_1, \lambda_2, \lambda_3, \dots)$ is a non-increasing sequence in $\mathbb{N}_0^\infty$ with finitely many non-zero terms.

It has **size** $|\lambda| = \sum_{k=1}^{\infty} \lambda_k$ and

length $\ell(\lambda) = \max\{k \mid \lambda_k > 0\}$.

E.g.



$\lambda_1 = 8$
$\lambda_2 = 4$
$\lambda_3 = 4$
$\lambda_4 = 1$
$\lambda_5 = 0$
$\vdots$

$|\lambda| = 17$

$\ell(\lambda) = 4$

Denote by $\mathbb{Y}_n$ the set of diag's with $|\lambda| = n$ and $\mathbb{Y}$ the set of all Young diag's.

<u>Def</u> The **Schur polynomial** in n variables parametrized by a Young diagram $\lambda \neq \emptyset$ with $\ell(\lambda) \leq n$ is:

$$s_\lambda(x_1, x_2, \dots, x_n) = \frac{\det \begin{bmatrix} x_1^{\lambda_1+n-1} & x_1^{\lambda_2+n-1} & \dots & x_1^{\lambda_n+n-1} \\ x_2^{\lambda_1+n-2} & x_2^{\lambda_2+n-2} & \dots & x_2^{\lambda_n+n-2} \\ \vdots & & & \\ x_n^{\lambda_1} & x_n^{\lambda_2} & \dots & x_n^{\lambda_n} \end{bmatrix}}{\det \begin{bmatrix} x_1^{n-1} & x_1^{n-1} & \dots & x_1^{n-1} \\ x_2^{n-2} & x_2^{n-2} & \dots & x_2^{n-2} \\ \vdots & & & \\ 1 & 1 & \dots & 1 \end{bmatrix}}$$

skew-symmetrization of $x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n}$

$$= \frac{\det \left[x_i^{\lambda_j+n-i} \right]_{i,j=1,\dots,n}}{\prod_{i<j} (x_i - x_j)}$$

This is a polynomial. (Why?) It is symmetric. (Why?)

Equivalently, the Jacobi-Trudi identities: for $\lambda \in \mathbb{Y}_n$,

$$S_\lambda(x_1,\dots,x_n) = \det \begin{bmatrix} h_{\lambda_1}(x_1,\dots,x_n) & h_{\lambda_1+1}(x_1,\dots,x_n) & \cdots & h_{\lambda_1-1+n}(x_1,\dots,x_n) \\ h_{\lambda_2-1}(x_1,\dots,x_n) & h_{\lambda_2}(x_1,\dots,x_n) & \cdots & h_{\lambda_2-2+n}(x_1,\dots,x_n) \\ \vdots & & & \\ h_{\lambda-n+1}(x_1,\dots,x_n) & h_{\lambda_n-n+2}(x_1,\dots,x_n) & \cdots & h_{\lambda_n}(x_1,\dots,x_n) \end{bmatrix}$$

(Exercise. Also available in the basis of e_n, p_n. See Macdonald.)

To each $\lambda \in \mathbb{Y}$, we will associate the fps

$$S_\lambda = S_\lambda(x_1, x_2, \dots) \quad (x_1, x_2, \dots)$$

$$= \det \begin{bmatrix} h_{\lambda_1}(x_1,x_2,\dots) & h_{\lambda_1+1}(x_1,x_2,\dots) & \cdots & h_{\lambda_1-1+n}(x_1,x_2,\dots) \\ h_{\lambda_2-1}(x_1,x_2,\dots) & h_{\lambda_2}(x_1,x_2,\dots) & \cdots & h_{\lambda_2-2+n}(x_1,x_2,\dots) \\ \vdots & & & \\ h_{\lambda-\ell(\lambda)+1}(x_1,x_2,\dots) & h_{\lambda_n-\ell(\lambda)+2}(x_1,x_2,\dots) & \cdots & h_{\lambda_{\ell(\lambda)}}(x_1,x_2,\dots) \end{bmatrix}$$

Frequently asked positivity questions:

(1) When does a symmetric function expand positively in a given basis?

E.g. Thm (Kostka) $s_\lambda(x_1, x_2, \dots, x_n) = \sum_{\mu \in Y_n} \underbrace{K_{\lambda,\mu}}_{\substack{\text{Combinatorial} \\ \geq 0}} \underbrace{m_\mu(x_1, x_2, \dots, x_n)}_{\text{monomials}}$

(Exercise)

E.g. When does a symmetric function expand positively in the Schur basis?

(Consequences for algebraic geometry)

(2) $s_\lambda(x_1, \dots, x_n)$ "evaluated" on $\mathbb{R}^n_{\geq 0}$ is ≥ 0. (Substitution $x_i \mapsto a_i$, $i=1,\dots,n$)

More generally:

Def A **specialization** is an algebra homomorphism $\rho: \Lambda \to \mathbb{C}$, $f \mapsto f(\rho)$

i.e. $(f+g)(\rho) = f(\rho) + g(\rho)$, $(\alpha f)(\rho) = \alpha f(\rho)$

$(fg)(\rho) = f(\rho)\, g(\rho) \qquad \forall\, \alpha \in \mathbb{C},\ f, g \in \Lambda$

$\rho : \Lambda \to \mathbb{C}, \quad f \mapsto f(\rho)$

Notice: ρ is defined by its values on elements of an alg. basis of Λ.

(Can view ρ as any element of $\mathbb{C}^\infty$.)

(Exercise: is any substitution $x_i \mapsto a_i \in \mathbb{C}$ an evaluation?)

<u>Def</u> ρ is Schur-positive if $s_\lambda(\rho) \geq 0 \quad \forall \lambda \in \mathbb{Y}$

Allows one to define a measure:

<u>Def</u> (Okounkov) Let $\rho_1, \rho_2 : \Lambda \to \mathbb{C}$ be two Schur-positive specializations s.t. $\sum_{\lambda \in \Lambda} s_\lambda(\rho_1) s_\lambda(\rho_2) < \infty$.

The Schur measure on $\mathbb{Y}$ is the probability measure

$$\mathbb{P}_{\rho_1, \rho_2}(\lambda) \propto s_\lambda(\rho_1) s_\lambda(\rho_2)$$

See Borodin-Gorin lecture notes for examples, also properties / uses of Schur measures.

Also for the definition of the Schur process, a prob measure on $\lambda^{(1)}, \mu^{(1)}, \lambda^{(2)}, \mu^{(2)}, \dots, \lambda^{(N)}, \mu^{(N)} \in \Lambda$ parametrized by $2N$ Schur-positive spec.'s $\rho_0^+, \dots, \rho_{N-1}^+, \rho_1^-, \dots, \rho_N^-$

$\rightsquigarrow$ general framework giving models that can be analyzed

e.g. measure on plane partitions $\propto q^{\mathrm{vol}(\pi)}$, long range TASEP

When is a spec. $\rho : \Lambda \to \mathbb{C}$ Schur positive?

Recall: $$s_\lambda = \det \begin{bmatrix} h_{\lambda_1} & h_{\lambda_1+1} & \cdots & h_{\lambda_1+\ell(\lambda)-1} \\ h_{\lambda_2-1} & h_{\lambda_2} & \cdots & h_{\lambda_2+\ell(\lambda)-2} \\ \vdots & \vdots & & \vdots \\ h_{\lambda_{\ell(\lambda)}-\ell(\lambda)+1} & h_{\lambda_{\ell(\lambda)}-\ell(\lambda)+2} & \cdots & h_{\lambda_{\ell(\lambda)}} \end{bmatrix}$$

ρ is Schur positive $\iff$ $h_1(\rho), h_2(\rho), \ldots$ is Toeplitz totally positive

<u>Corollary</u> Schur-positive specializations are parametrized by

$\alpha \geq 0$, $\beta_1 \geq \beta_2 \geq \ldots \geq 0$, $\gamma_1 \geq \gamma_2 \geq \ldots \geq 0$ s.t. $\sum_i \beta_i + \sum_i \gamma_i < \infty$

and given by
$$\sum_{n \geq 0} h_n(\rho_{\alpha,\beta,\gamma})\, z^n = e^{z\alpha} \prod_{i \geq 0} \frac{(1+\beta_i z)}{(1-\gamma_i z)}.$$

(Analogue for Macdonald functions conj' Kerov '92, proof Matveev '17)

Part II - A moment sequencer's toolkit

Recall:

Algebraic/combinatorial object $\rightsquigarrow$ Representing matrix $\rightsquigarrow$ Positivity of the matrix

Prototypical example in probability:

combinatorial seq $(a_n)_{n\geq 0}$ $(a_0 = 1)$

When is there some prob. measure μ on $\mathbb{R}$ s.t. $\forall n \in \mathbb{N}$

$$a_n = \int_{\mathbb{R}} x^n \, d\mu(x) \; ?$$

Hamburger: positive semi-definiteness of the Hankel matrices $[a_{i+j}]_{i,j\geq 0}$

(1) Continued fractions

Ex. 1 Let $\varphi_0 = 1,\ \varphi_1 = 2,\ \varphi_n = 1 + \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{\cdots\, 2}}}}$ } depth: n levels

$1, 2, 3/2, 5/3, 8/5, \dots = 1+0, 1+1, 1+1/2, 1+2/3, 1+3/5, \dots$

Exercise: Show that

- $\varphi_n = 1 + \dfrac{f_n}{f_{n+1}}$ where $f_0 = 0,\ f_1 = 1,\ f_n = f_{n-1} + f_{n-2}$

  $0, 1, 1, 2, 3, 5, 8, \dots$

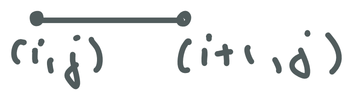
- $\varphi_n \to \varphi = \dfrac{1+\sqrt{5}}{2}$

Ex. 2 As formal power series

$$\cfrac{1}{1-\cfrac{z}{1-\cfrac{2z}{1-\cfrac{3z}{\cdots}}}} = \sum_{n\geq 0} \underbrace{(2n-1)(2n-3)\cdots 3\cdot 1}_{=:\,(2n-1)!!}\, z^n \qquad \text{(Euler)}$$

Def

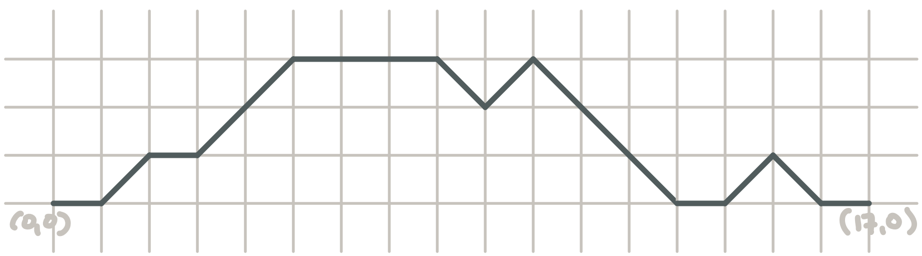
A **Motzkin path** of length n is a walk in $\mathbb{N}_0 \times \mathbb{N}_0$ that starts at $(0,0)$, ends at $(n,0)$, consists of:

- level steps 

- upsteps 

- downsteps 

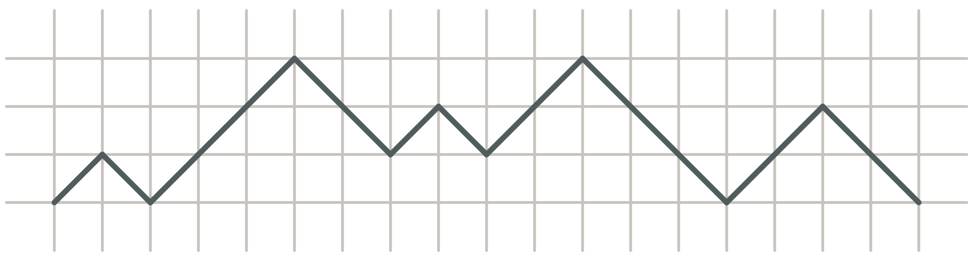
and remains positive (i.e. $j \geq 0$ at each step).



Exercise: Let $m_n = \#$ Motzkin paths with n steps

Then $$\sum_{n \geq 0} m_n z^n = \frac{1-z\sqrt{1-2z-3z^2}}{2z^2}$$

Def A **Dyck path** is a Motzkin path consisting only of upsteps and downsteps.



Exercise: # Dyck paths with n steps $= C_n := \frac{1}{n+1}\binom{2n}{n}$

Recall: $R[[X]]$ denotes the set of formal power series

$$\sum_{k\geq 0} a_k x^k$$ with $a_0, a_1, \ldots \in R$ equipped with addition:

$$\sum_k a_k x^k + \sum_n b_k x^k = \sum_k (a_k + b_k) x^k$$

and multiplication $$\left(\sum_k a_k x_k\right)\left(\sum_k b_k x_k\right) = \sum_k c_k x^k$$

where $$c_k = \sum_{l=0}^{k} a_l b_{k-l} .$$

The topology on $R[[X]]$ is the product topology, with the discrete topology on R.

i.e. $$\sum_k a_k^{(n)} x^k \xrightarrow[n\to\infty]{} \sum_k a_k x^k$$ iff $a_k^{(n)} = a_k$ for n large enough

From now on, take

$R = \mathbb{C}$ or

$R =$ quotients of multivariate polynomials

(when working with "combinatorial statistics")

Recall: When $a_0 \neq 0$, $\sum_k a_k x^k$ has a multiplicative inverse.

E.g. $\dfrac{1}{1-x} = \sum_{k \geq 0} x^k$

<u>Thm</u> (Flajolet '80)

The sequence

$$\left(\cfrac{1}{1-\alpha_0 z - \cfrac{\beta_1 z^2}{1-\alpha_1 z - \cfrac{\beta_2 z^2}{\ddots\, 1-\alpha_{n-1} z - \beta_n z^2}}} \right)_{n \geq 0}$$

converges as formal power series. Its limit is denoted:

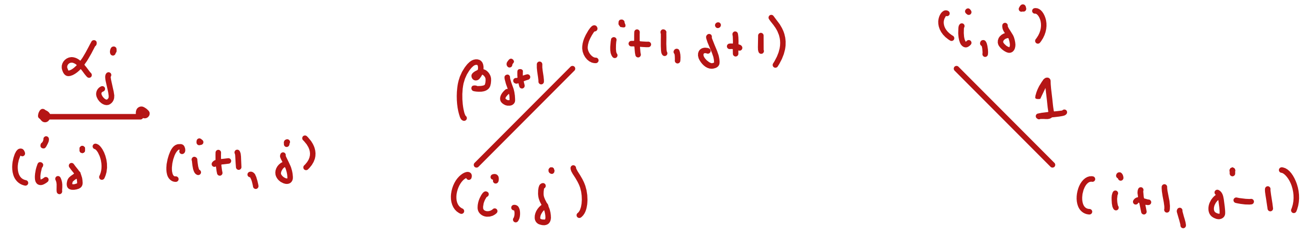
$$\cfrac{1}{1-\alpha_0 z - \cfrac{\beta_1 z^2}{1-\alpha_1 z - \cfrac{\beta_2 z^2}{\cdots}}}$$

(Continued ...)

Thm (Flajolet '80) ... Moreover

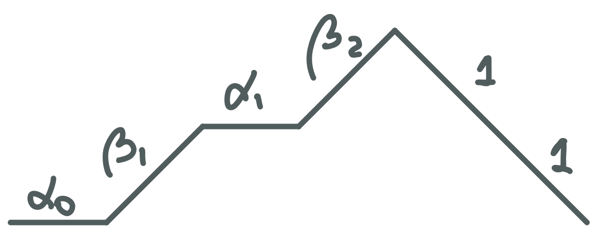
$$\cfrac{1}{1-\alpha_0 z - \cfrac{\beta_1 z^2}{1-\alpha_1 z - \cfrac{\beta_2 z^2}{\cdots}}} = \sum_{n\geq 0} \sum_{m\in\mathcal{M}(n)} wt(m)\, z^n$$

where $\mathcal{M}(n)$ is the set of Motzkin paths with n steps labeled as:



with $wt(m)$ = product of the labels.

Example:



$m\in\mathcal{M}(6)$

$wt(m) = \alpha_0\beta_1\alpha_1\beta_2$

Proof by example: Start expanding

$$C_n(z) = \cfrac{1}{1-\alpha_0 z - \cfrac{\beta_1 z^2}{1-\alpha_1 z - \cfrac{\beta_2 z^2}{\cdots}}} = \frac{1}{1-\alpha_0 z - \beta_1 z^2 C'(z)}$$

$$= 1 + \left(\alpha_0 z + \beta_1 z^2 C'(z)\right) + \left(\alpha_0 z + \beta_1 z^2 C'(z)\right)^2 + \left(\cdots\right)^3 + \cdots$$

$$[z^0]\; C_n(z) = 1$$

$$[z]\; C_n(z) = \alpha_0$$

$$[z^2]\; C_n(z) = \alpha_0^2 + \beta_1 [z^0]\, C'(z) = \alpha_0^2 + \beta_1$$

$$[z^3]\; C_n(z) = \alpha_0^3 + 2\alpha_0\beta_1 [z^0]\, C'(z) + \beta_1 [z]\, C'(z)$$

$$= \alpha_0^3 + 2\alpha_0\beta_1 + \beta_1\alpha_1$$

Exercise: write down a proof

More examples:

$$\cfrac{1}{1-\cfrac{z}{1-\cfrac{2z}{1-\cfrac{3z}{\cdots}}}} = \sum_{n\geq 0} (2n-1)(2n-3)\cdots 3\cdot 1 \; z^n$$

$$\cfrac{1}{1-\cfrac{z}{1-\cfrac{z}{1-\cfrac{z}{\cdots}}}} = \sum_{n\geq 0} C_n \, z^n$$

$$\cfrac{1}{1-z-\cfrac{z^2}{1-2z-\cfrac{2z^2}{\cdots}}} = \sum_{n\geq 0} B_n \, z^{2n}$$

$$\cfrac{1}{1-z-\cfrac{1^2\cdot z^2}{1-\underbrace{3}_{2n+1}z-\cfrac{\underbrace{2^2}_{n^2}\cdot z^2}{\cdots}}} = \sum_{n\geq 0} n! \; z^{2n}$$

Q: What does $\alpha_n \geq 0$, $\beta_n \geq 0$ imply?

Next : orthogonal polynomials