Positivity everywhere

Lecture 3

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Last time: Formal power series and Continued fractions

$$\cfrac{1}{1-\cfrac{z}{1-\cfrac{2z}{1-\cfrac{3z}{\cdots}}}} = \sum_{n\geq 0} (2n-1)(2n-3)\cdots 3\cdot 1 \; z^n$$

$$\cfrac{1}{1-\cfrac{z}{1-\cfrac{z}{1-\cfrac{z}{\cdots}}}} = \sum_{n\geq 0} C_n \, z^n$$

$$\cfrac{1}{1-z-\cfrac{z^2}{1-2z-\cfrac{2z^2}{\cdots}}} = \sum_{n\geq 0} B_n \, z^n$$

$$\cfrac{1}{1-z-\cfrac{1^2\cdot z^2}{1-\underbrace{3}_{2n+1}z-\cfrac{\underbrace{2^2}_{n^2}\cdot z^2}{\cdots}}} = \sum_{n\geq 0} n! \, z^{2n}$$

(2) Orthogonal Polynomials

Start w/ $(a_n)_{n\geq 0}$ real seq.

Define linear functional $\mathcal{L}$ on $\mathbb{C}[x]$ by $\mathcal{L}(x^n) = a_n$

Observe:

for $p(x) = c_0 + c_1 x + \cdots + c_n x^n$,

$$\mathcal{L}(p\bar{p}) = \mathcal{L}\left((c_0 + c_1 x + \cdots + c_n x^n)(\bar{c}_0 + \bar{c}_1 x + \cdots + \bar{c}_n x^n) \right)$$

$$= \sum_{j,\ell=0}^{n} a_{j+\ell}\, c_j \bar{c}_\ell \geq 0$$

when (a_n) is "positive definite"

Define a (semi-)inner product on $\mathbb{C}[x]$ by

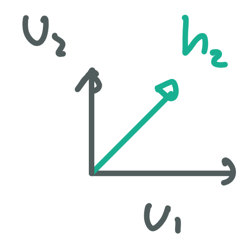
$$\langle p, r \rangle = L(p\bar{r}).$$

Monomials $h_0 = 1,\ h_1 = x,\ h_2 = x^2, \ldots$

are lin. indep. but <u>not</u> orthogonal. ... Orthogonalize!

Gram-Schmidt:

$u_0 = h_0 = 1 \qquad e_0 = 1$



$$u_1 = h_1 - \langle h_1, e_0 \rangle e_0 = x - L(x \cdot 1) = x - a_1, \quad e_1 = \frac{x - a_1}{\sqrt{a_2 - a_1^2}}$$

$$u_2 = h_2 - \langle h_2, e_0 \rangle e_0 - \langle h_2, e_1 \rangle e_1$$

$$= x^2 - \langle x^2, 1 \rangle 1 - \langle x^2, \frac{x - a_1}{\sqrt{a_2 - a_1^2}} \rangle \frac{(x - a_1)}{\sqrt{a_2 - a_1^2}}$$

$$= x^2 - x\frac{a_3 - a_2 a_1}{(a_2 - a_1^2)} - a_2 - \frac{a_2 a_1 - a_3}{(a_2 - a_1^2)} a_1 = x^2 - x\frac{a_3 - a_2 a_1}{(a_2 - a_1^2)} + \frac{a_1 a_3 - a_2^2}{(a_2 - a_1^2)}$$

Example : $a_0 = 1$, $\quad a_n = \begin{cases} 0 & , \ n \text{ odd} \\ (n-1)!! & , \ n \text{ even} \end{cases}$

$a_2 = 1$, 0, $a_4 = 3$, 0, $a_6 = 15$, $\ldots$

$L(x^n) = a_n$.

Orthogonalize $1, x, x^2, \ldots$

$U_0 = 1$

$U_1 = x - \langle x, 1 \rangle 1 = x - \cancel{L(x)}^{0}$

$U_2 = x^2 - \langle x^2, 1 \rangle 1 - \dfrac{\langle x^2, x \rangle}{\langle x, x \rangle} x$

$\quad = x^2 - 1 - 0 = x^2 - 1$

$\vdots$

Hermite OPS

$1, x, x^2 - 1, \ldots$

(we will further refine these)

Gram-Schmidt: $\quad u_0 = h_0$

$$u_n = \frac{\begin{vmatrix} \langle h_0, h_0\rangle & \langle h_1, h_0\rangle & \cdots & \langle h_{n-1}, h_0\rangle & \langle h_n, h_0\rangle \\ \langle h_0, h_1\rangle & \langle h_1, h_1\rangle & \cdots & \langle h_{n-1}, h_1\rangle & \langle h_{n-1}, h_1\rangle \\ \vdots & \vdots & & \vdots & \vdots \\ h_0 & h_1 & \cdots & h_{n-1} & h_n \end{vmatrix}}{\begin{vmatrix} \langle h_0, h_0\rangle & \cdots & \langle h_{n-1}, h_0\rangle \\ \langle h_0, h_1\rangle & \cdots & \langle h_{n-1}, h_1\rangle \\ \vdots & & \vdots \\ \langle h_0, h_{n-1}\rangle & \cdots & \langle h_{n-1}, h_{n-1}\rangle \end{vmatrix}}$$

(Exercise)

$h_0 = 1, \quad h_1 = x, \quad h_2 = x^2, \ldots \qquad\qquad L(x^n) = a_n$

$$U_n = \begin{vmatrix} 1 & a_1 & a_2 & \ldots & a_n \\ a_1 & a_2 & a_3 & \ldots & a_{n+1} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n-1} & a_{n-2} & a_{n-3} & \ldots & a_{2n-1} \\ 1 & x & x^2 & \ldots & x^n \end{vmatrix} \quad \frac{1}{\det H_{n-1}}$$

$$U_0 = 1, \quad U_1 = x - a_1, \quad U_2 = x^2 - x\,\frac{a_3 - a_2 a_1}{(a_2 - a_1^2)} + \frac{a_1 a_3 - a_2^2}{(a_2 - a_1^2)}, \quad \ldots$$

are orthogonal wrt L, i.e. $L(U_j U_k) = \delta_{jk} L(|U_j|^2)$

<u>Thm</u> Polys $(U_n)_{n\geq 1}$ orthogonal wrt some $L \geq 0$ satisfy

$$U_{n+1}(x) = (x - \alpha_n)\, U_n(x) - \beta_n\, U_{n-1}(x)$$

where $\alpha_n, \beta_n \in \mathbb{R}$ and $\beta_n \geq 0$ $\forall n$.

Exercise: Prove thm & deduce formulas for α_n and β_n.

In particular, show that $\beta_n = \dfrac{\det(H_{n+1})\det(H_{n-1})}{\det^2(H_n)}$

$(n \geq 2)$

Example: $a_n = \begin{cases} 0 & n \text{ odd} \\ (n-1)!! & n \text{ even} \end{cases}$

$U_0 = 1$, $U_1 = x$, $U_2 = x^2 - 1$, ... Hermite OPS

$$U_{n+1}(x) = x\, U_n(x) - n\, U_{n-1}(x)$$

$\alpha_n, \beta_n \in \mathbb{C}$

<u>Thm</u> (Stieltjes, Shohat, Stone, Favard, ...)

Polys $(U_n)_{n\geq 1}$ satisfying

$$U_{n+1}(x) = (x-\alpha_n)\, U_n(x) - \beta_n\, U_{n-1}(x)$$

are orthogonal wrt unique lin funct L on $\mathbb{C}[x]$.

$$L \geq 0 \iff \alpha_n, \beta_n \in \mathbb{R} \text{ and } \beta_n \geq 0 \;\; \forall n.$$

Proof: Exercise

Started with $(a_n)_{n\geq 0}$ strictly positive definite

Defined a positive linear funct. on $\mathbb{C}[x]$ by $L(x^n) = a_n$

Defined inner product on $\mathbb{C}[x]$ by

$$\langle p, q \rangle = L(p\bar{q})$$

let $\mathcal{H} = \overline{\mathbb{C}[x]}^{\langle,\rangle}$.

Gave an orthonormal basis $\mathcal{U} = \{ U_n / \|U_n\| \}_{n\geq 0}$

Define linear operator M on $\mathrm{span}(\mathcal{U})$ as $M\,p(x) = x\,p(x)$

Observe that $\langle M^n \cancel{U_0}', \cancel{U_0}' \rangle = a_n = L(x^n)$

Lift to a symmetric op. on $\mathcal{H}$, self-adjoint extension

$\rightsquigarrow$ spectral thm supplies $\mu \geq 0$ s.t. $a_n = \int_{\mathbb{R}} x^n \, d\mu(x)$

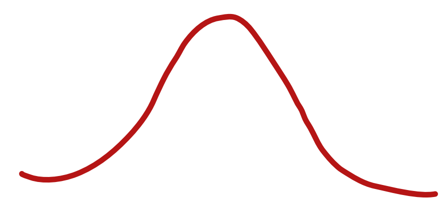
Example: $a_0 = 1$, $a_n = \begin{cases} 0, & n \text{ odd} \\ (n-1)!!, & n \text{ even} \end{cases}$

$a_2 = 1$, $a_4 = 3$, $a_6 = 15$, ...

counting?

$U_0(x) = 1$, $U_1(x) = x$, $U_{n+1} = x U_n - n U_{n-1}$

$$a_n = \int_{\mathbb{R}} x^n \frac{e^{-x^2/2}}{\sqrt{2\pi}} \, dx$$



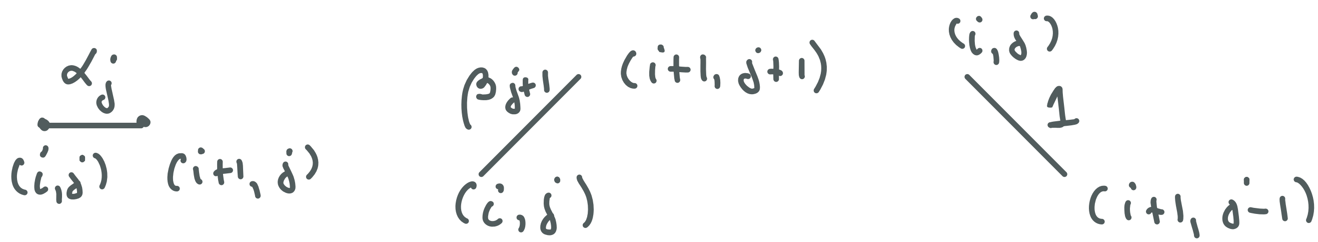
Generally: questions of uniqueness (not relevant this time)

$U_0(x)=1$, $U_1(x)=x-\alpha_1$, $U_{n+1}(x) = (x-\alpha_n)\,U_n(x) - \beta_n\, U_{n-1}(x)$

Thm (Viennot '84) For $n,k,\ell \in \mathbb{N}_0$

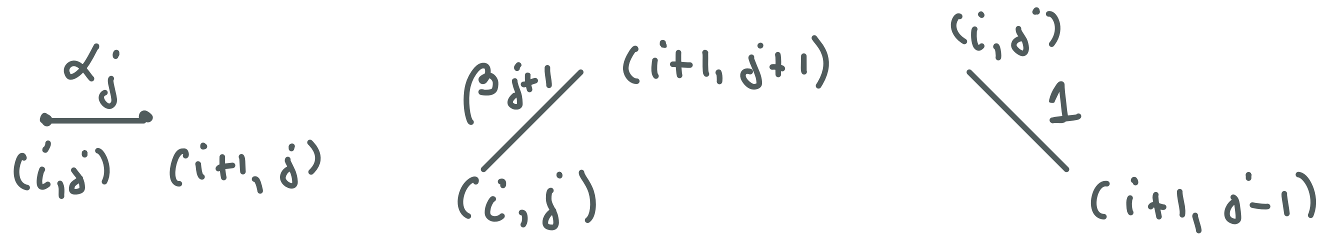
$$L\left(x^n U_k(x)\, U_\ell(x)\right) = \begin{cases} \beta_1\beta_2\cdots\beta_k \displaystyle\sum_{w\in \mathcal{M}_{k,\ell,n}} wt(w) & n\neq 0 \\ \beta_1\beta_2\cdots\beta_k\, \delta_{\ell k} & n=0 \end{cases}$$

where $\mathcal{M}_{k,\ell,n}$ is the set of "Motzkin" paths starting at $(0,k)$ and ending at (n,ℓ) with weights



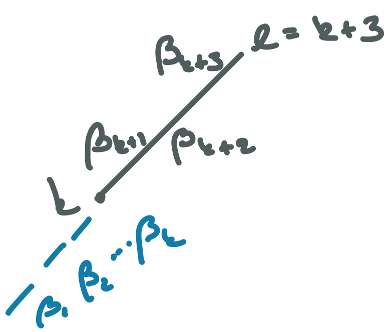
$$L(x^n U_k(x) U_\ell(x)) = \beta_1 \beta_2 \cdots \beta_k \sum_{w \in \mathcal{M}_{k,\ell,n}} wt(w)$$

where $\mathcal{M}_{k,\ell,n}$ is the set of "Motzkin" paths starting at $(0,k)$ and ending at (n,ℓ) with weights

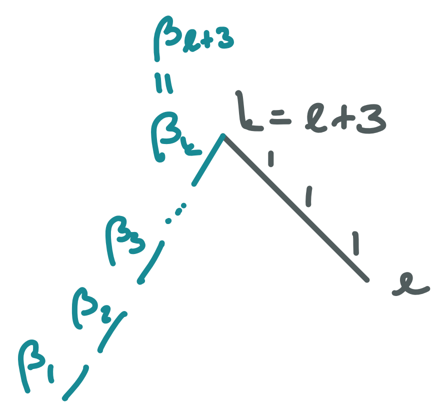


Notice: Symmetry in k and ℓ

$k<\ell$

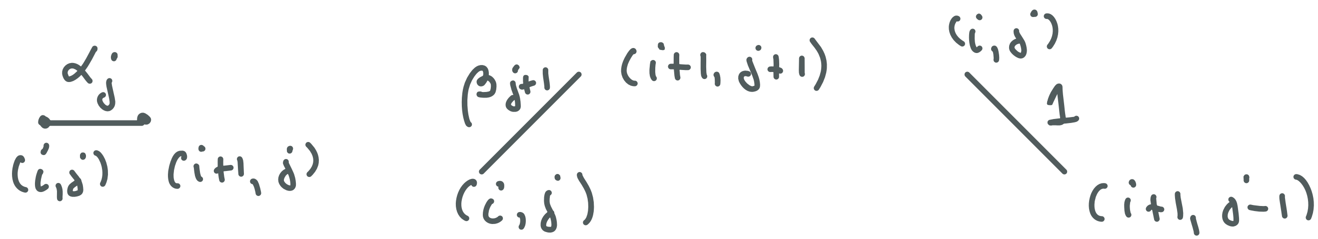


$k>\ell$



$$\mathcal{L}\left(x^n U_k(x) U_\ell(x)\right) = \beta_1 \beta_2 \cdots \beta_k \sum_{w \in \mathcal{M}_{k,\ell,n}} wt(w)$$

where $\mathcal{M}_{k,\ell,n}$ is the set of "Motzkin" paths starting at $(0,k)$ and ending at (n,ℓ) with weights



Special case $k=\ell=0$: $\quad \mathcal{L}(x^n) = \sum_{w \in \mathcal{M}_n} wt(w)$

Hence

$$\sum_{n \geq 0} \mathcal{L}(x^n) z^n = \cfrac{1}{1 - \alpha_0 z - \cfrac{\beta_1 z^2}{1 - \alpha_1 z - \cfrac{\beta_2 z^2}{\cdots}}}$$

$U_0(x) = 1$, $U_1(x) = x - \alpha_1$, $U_{n+1}(x) = (x - \alpha_n) U_n(x) - \beta_n U_{n-1}(x)$

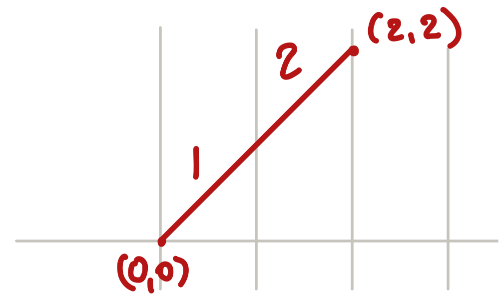
$$U_{n+1}(x) = x\,U_n(x) - n\,U_{n-1}(x)$$

Example: Hermite OPs

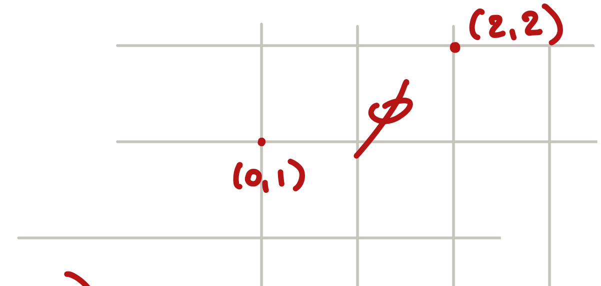
$$1,\ x,\ x^2-1,\ x^3-3x,\ \dots$$

$$L(x^2 U_0 U_2) = L(x^2\cdot 1\cdot (x^2-1))$$

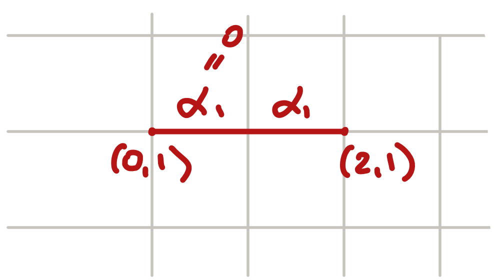
$$= 3!! - 1!! = 2 = 2$$

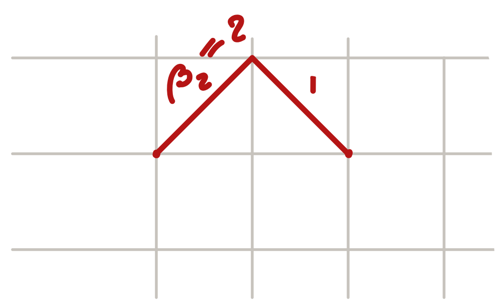


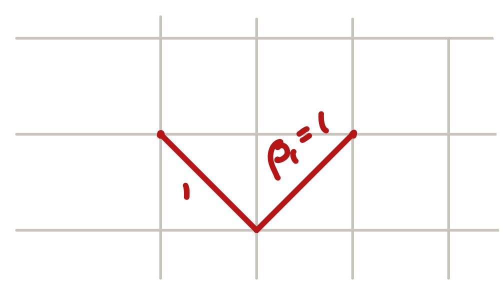
$$L(x^2 U_1 U_2) = L\left(x^2\cdot x\cdot (x^2-1)\right) = 0$$



$$L(x^2 U_1^2) = L(x^4) = 3!! = 3 = \beta_1(1+2)$$





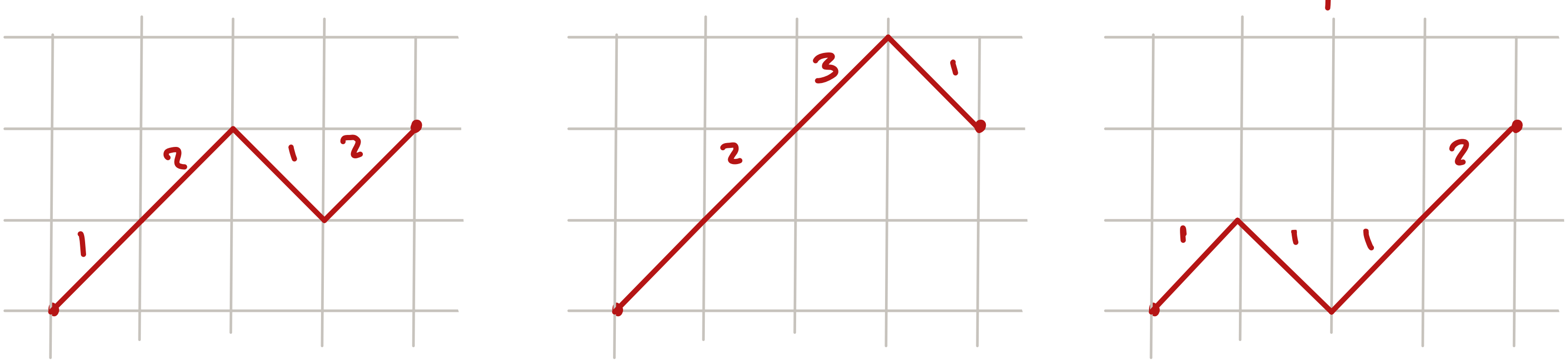


$$L(x^2 U_2^2) = L\left(x^2(x^2-1)(x^2-1)\right) = L(x^6 - 2x^4 + x^2)$$

$$= 5!! - 2\cdot 3!! + 1!! = 15 - 6 + 1 = 10 = \beta_1 \beta_2 (3+2)$$

$$L(x^4 U_0 U_2) = L\left(x^4(x^2-1)\right) = 5!! - 3!! = 15 - 3 = 12$$

$$= \underset{1}{\cancel{\beta_0}}(4+6+2)$$



Putting (1) and (2) together:

Consider a sequence $(a_n)_{n\ge 0}$, $a_0=1$. "J-fraction"

Expand its generating function as $\displaystyle\sum_{n\ge 0} a_n z^n = \cfrac{1}{1-\alpha_0 z - \cfrac{\beta_1 z^2}{1-\alpha_1 z - \cfrac{\beta_2 z^2}{\cdots}}}$

Equivalently, $(a_n)_{n\ge 0}$ is the sequence of moments of the orthogonalizing functional L for the polynomials

$$U_0(x)=1, \quad U_1(x)=x-1, \quad U_{n+1}(x)=(x-\alpha_n)\,U_n(x)-\beta_n U_{n-1}(x)$$

We have $L \geq 0$

$\iff (a_n)_{n\geq 0}$ is a seq. of moments of a probability measure on $\mathbb{R}$

$\iff$ $\alpha_n, \beta_n \in \mathbb{R}$ and either $\beta_n > 0 \ \forall n$ (measure has infinite support)

or

$\beta_n > 0 \ \forall \ n \leq N$ and C.F. terminates with β_N

(measure supported on N elements)

$\iff$ matrices $H_1, H_2, H_3, \dots$ are positive semidefinite.

(Hamburger moment problem)

What about total positivity?

$H = [a_{i+j}]_{i,j\geq 0}$ totally positive $\iff$ measure supported on $[0,\infty)$

(Stieltjes moment problem)

$$\iff \sum_{n\geq 0} a_n z^n = \cfrac{1}{1 - \cfrac{\beta_1 z}{1 - \cfrac{\beta_2 z}{\cdots}}}$$ with β_n as above.

(S-fraction)

More examples:

$$\cfrac{1}{1-\cfrac{z}{1-\cfrac{2z}{1-\cfrac{3z}{\cdots}}}} = \sum_{n\geq 0} (2n-1)(2n-3)\cdots 3\cdot 1 \; z^n$$

$\beta_n = n$

$$\cfrac{1}{1-\cfrac{z}{1-\cfrac{z}{1-\cfrac{z}{\cdots}}}} = \sum_{n\geq 0} C_n z^n$$

$\beta_n = 1$

$$\cfrac{1}{1-z-\cfrac{z^2}{1-2z-\cfrac{2z^2}{\cdots}}} = \sum_{n\geq 0} B_n z^n$$

$\beta_n = n$

$$\cfrac{1}{1-z-\cfrac{1^2\cdot z^2}{1-\underbrace{3}_{2n+1}z-\cfrac{\underbrace{2^2}_{n^2}\cdot z^2}{\cdots}}} = \sum_{n\geq 0} n!\, z^n$$

$\beta_n = n^2$

(3) Operator models

Let $\mathcal{H}$ be $\mathbb{C}$-Hilbert with o.n. basis $(e_n)_{n\geq 0}$.

Let A and $\tilde{A}$ be linear operators with matrices in (e_n):

$$A = \begin{bmatrix} \alpha_0 & \beta_1 & & & \\ 1 & \alpha_1 & \beta_2 & & \\ & 1 & \alpha_2 & \beta_3 & \\ & & 1 & \ddots & \ddots \\ & & & \ddots & \end{bmatrix}, \quad \tilde{A} = \begin{bmatrix} \alpha_0 & \sqrt{\beta_1} & & & \\ \sqrt{\beta_1} & \alpha_1 & \sqrt{\beta_2} & & \\ & \sqrt{\beta_2} & \alpha_2 & \sqrt{\beta_3} & \\ & & \ddots & \ddots & \ddots \\ & & & \ddots & \end{bmatrix}$$

$\alpha_0, \alpha_1, \ldots \in \mathbb{R}$

$\beta_1, \beta_2, \ldots \geq 0$

Observe: $$\sum_{m \in \mathcal{M}_n} wt(m) = \langle A^n e_0, e_0 \rangle$$

$$= \langle \tilde{A}^n e_0, e_0 \rangle$$

n^{th} moment of $\tilde{A}$ wrt $\mathbb{E}(\cdot) = \langle \cdot\, e_0, e_0 \rangle$

$$\left((A^n)_{1,1} = \sum a_{1 i_1} a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_k i_{k+1}} \cdots a_{i_{n-1} 1} \right)$$

$$a_{i_k i_{k+1}} = \begin{cases} \xrightarrow{\ \alpha_{i_k}\ } & \text{when } i_{k+1} = i_k \\ \beta_{i_{k+1}} \nearrow & \text{when } i_{k+1} = i_k + 1 \\ \searrow 1 & \text{when } i_{k+1} = i_k - 1 \end{cases}$$

Def A noncommutative probability space is a pair $(\mathcal{A}, \varphi)$ where:

- $\mathcal{A}$ is a $*$-algebra, $1 \in \mathcal{A}$. "Noncommutative random variables"
- $\varphi : \mathcal{A} \to \mathbb{C}$ linear, $\varphi(1) = 1$, $\varphi(x^* x) \geq 0 \ \forall x \in \mathcal{A}$. "Expectation"

Example 1: $\mathcal{A} = \bigcap_{p \geq 0} L^p_{\mathbb{C}}(\Omega, \mathbb{P})$, $\varphi = \mathbb{E}$

Example 2: $\mathcal{A} = \mathrm{Mat}_{n \times n}(\mathbb{C})$, $\varphi = \frac{1}{n} \mathrm{Tr}$

Example 3: Combine Ex 1 & Ex 2 (Exercise)

Compare Def to Ex 1-3. Typically, $\mathcal{A}$ has more structure.

Def The distribution of $x \in \mathcal{A}$ is determined by its moments

$$\left\{ \varphi(\underbrace{x^{n_1}(x^*)^{m_1} x^{n_2}(x^*)^{m_2} \cdots x^{n_k}(x^*)^{n_k}}_{\text{interval partitions}}) : k \in \mathbb{N} \right\}$$

$n_1, \dots, n_k, m_1, \dots, m_k \in \mathbb{N}_0$

For $x, y \in \mathcal{A}$, $x = x^*$, $y = y^*$, their joint distribution is determined by:

$$\left\{ \varphi(x^{n_1} y^{m_1} x^{n_2} y^{m_2} \cdots x^{n_k} y^{m_k}) \right\}$$

$\rightsquigarrow$ Notions of independence = rules for factorizing moments

E.g. x, y classically independent $\Rightarrow \varphi(x y x^2 y) = \varphi(x^3)\varphi(y^2)$

all partitions

E.g. x, y Boolean independent $\Rightarrow \varphi(x y x^2 y)$

$$= \varphi(x)\varphi(x^2)(\varphi(y))^2$$

interval partitions

E.g. x, y freely independent $\Rightarrow$

$$\varphi\Big(\big(x-\varphi(x)\big)\big(y-\varphi(y)\big)\big(x^2-\varphi(x^2)\big)\big(y-\varphi(y)\big)\Big) = 0$$

Hence $\varphi(x\, y\, x^2\, y) = ?$ (Exercise)

non-crossing partitions

General observation: probabilistic structure

$\updownarrow$

combinatorial structure

Bona fide probability:

Suppose $\mathcal{A}$ is a C^* algebra. Take $x \in \mathcal{A}$ s.t. $x = x^*$.

By the Spectral Theorem:

$\exists\, \mu$ a prob. measure on $\mathbb{R}$ s.t. $\varphi(x^n) = \int_{\mathbb{R}} \xi^n \, d\mu(\xi)$

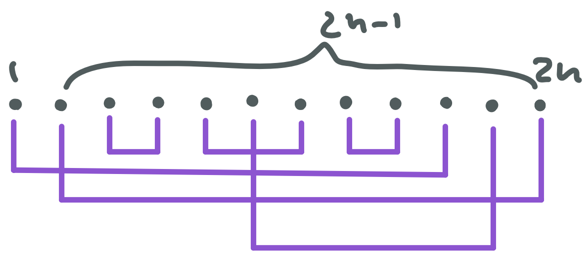
Classical	Free

Classical

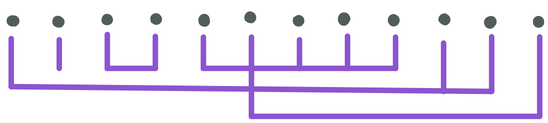
Central limit

$$\int_{\mathbb{R}} x^{2n} \frac{e^{-x^2/2}}{\sqrt{2\pi}} dx$$

$$= (2n-1)(2n-3)\cdots 5\cdot 3\cdot 1$$



Poisson limit

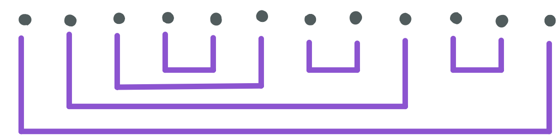


B_n

Free

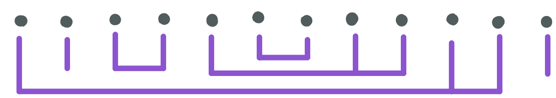
Central limit

$$\int_{[-2,2]} x^{2n} \frac{1}{2\pi}\sqrt{4-x^2}\, dx = \frac{1}{n+1}\binom{2n}{n}$$



C_n

Poisson limit



C_n (!)

Which types of combinatorial objects can play a structural role?

When is $(a_n)_{n\geq 0}$ a moment sequence? Next lect.

Positivity: Combinatorial factorization into irreducibles

vs.

moment - cumulant formula

Combinatorial view:

identify families of naturally occurring irreducibles into which objects can be decomposed and from which they can be uniquely reconstructed

JOURNAL OF COMBINATORIAL THEORY, Series A **38**, 143–169 (1985)

The Enumeration of Irreducible Combinatorial Objects

JANET SIMPSON BEISSINGER

University of Illinois at Chicago, Chicago, Illinois 60680

Communicated by the Managing Editors

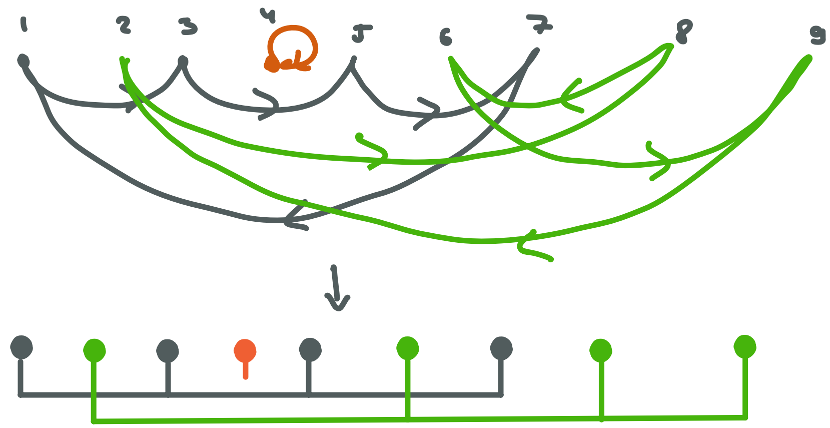
Received December 22, 1982

A general theory of irreducibility

Three canonical examples:

(ALL) Decomposing diagrams along all set partitions

E.g. permutations factorizing into cycles



each part is assigned a unique cycle

General formula (the exponential formula):

$$A(x) = \sum \frac{a_n}{n!} x^n \qquad \text{"alls"}$$

$$I(x) = \sum \frac{i_n}{n!} x^n \qquad \text{"irreducibles"}$$

$$A(x) = \exp(I(x)) \qquad \text{Riddell '51}$$

Three canonical examples:

(NC) Decomposing diagrams along non-crossing partitions

E.g. NC partitions themselves

E.g. Positroids decomposing into "connected positroids"

(recall Ardila, Rincón, Williams '16)

General formula (the exponential formula):

$$A(x) = \sum a_n x^n \quad \text{"alls"}$$

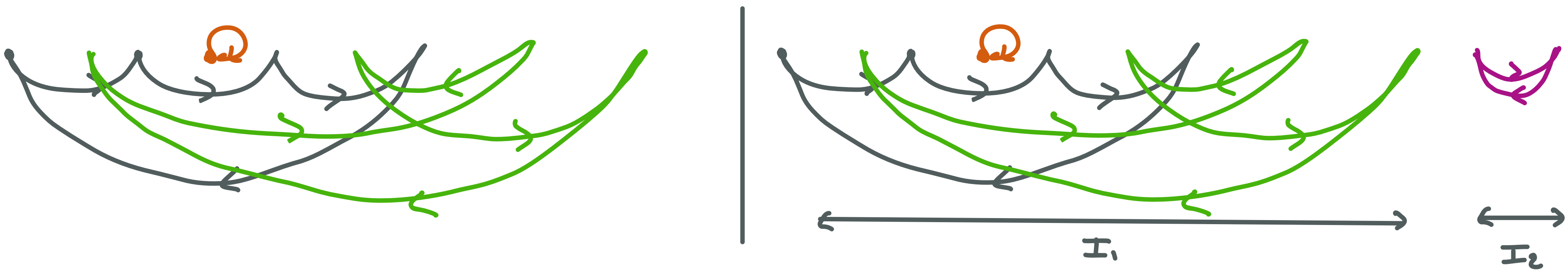
$$I(x) = \sum i_n x^n \quad \text{"irreducibles"}$$

$$A(x) = 1 + I(x A(x)) \qquad \text{Simpson Beissinger '85}$$

Three canonical examples:

(INTERVAL) Decomposing diagrams along interval partitions

E.g. Permutations factorizing along interval partitions into "Stabilized-interval-free" (SIF) permutations (Callan)



SIF

NOT SIF

General formula (the exponential formula):

$$A(x) = \sum a_n x^n \quad \text{"alls"}$$

$$I(x) = \sum i_n x^n \quad \text{"irreducibles"}$$

$$A(x) = (1 - I(x))^{-1}$$

Fact: when the sequence of "alls" is a moment sequences the sequence of "irreducibles" in the previous 3 examples are cumulant sequences.

Recall: cumulents linearize convolution

i.e. $K_{x+y} = K_x + K_y$

Dependent on the notion of independence

Specifically:

Classical moment-cumulant formula $M(z) = e^{K(z)}$

Free moment-cumulant formula $M(z) = 1 + C(zM(z))$ Speicher '94 (independent)

Boolean moment-cumulant formula $M(z) = \dfrac{1}{1 - I(z)}$

Next:

Which combinatorial structures are naturally captured through Motzkin paths (continued fractions)?

How can we unify a number of known combinatorial continued fractions?

How do we decompose a combinatorial statistic in terms of elementary building blocks?