Virginia integrable probability summer school 2024

Natasha Blitvic   Slim Kammoun

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We recall some definitions:

Hankel total positivity: A sequence $\boldsymbol{a} = (a_n)_{n\geq 0}$ is Hankel totally positive if all minors of the Hankel matrix $H = (a_{i+j})_{i,j\geq 0}$ are non-negative.

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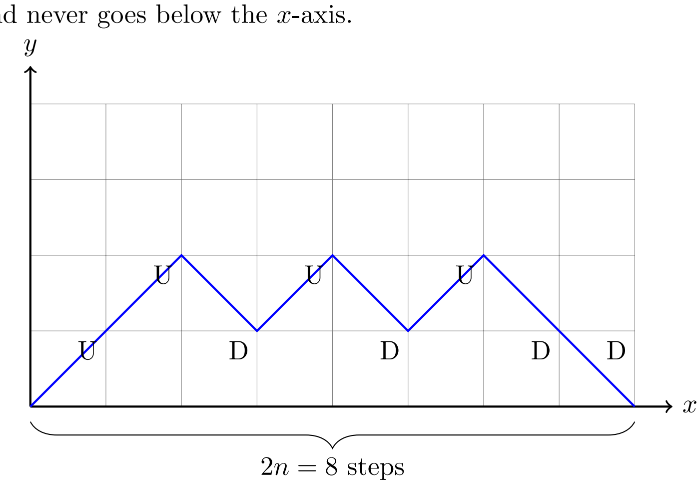
Toeplitz total positivity: A sequence $\boldsymbol{a} = (a_n)_{n\geq 0}$ is Toeplitz totally positive if all minors of the Hankel matrix $H = (a_{i-j})_{i,j\geq 0}$ are non-negative.

Moment Sequence: A sequence $\{a_n\}_{n=0}^{\infty}$ is called a moment sequence if there exists a positive Borel measure μ on $\mathbb{R}$ such that

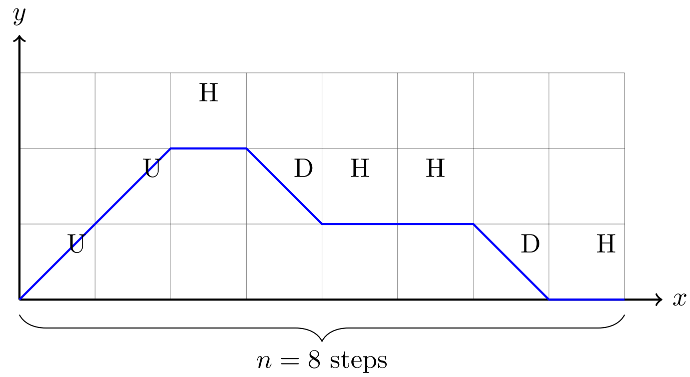
$$a_n = \int_{\mathbb{R}} x^n \, d\mu(x) \quad \text{for all } n \geq 0.$$

Dyck Path: A Dyck path is a lattice path in the plane with steps $U = (1, 1)$ (up) and $D = (1, -1)$ (down) that starts at the origin $(0, 0)$, ends at $(2n, 0)$, and never goes below the x-axis.



Dyck Numbers C_n: The number of Dyck paths of length $2n$.

Motzkin Path: A Motzkin path is a lattice path in the plane with steps $U = (1, 1)$ (up), $D = (1, -1)$ (down), and $H = (1, 0)$ (horizontal) that starts at the origin $(0,0)$, ends at $(n,0)$, and never goes below the x-axis.



Motzkin Numbers M_n: The number of Motzkin paths of length n.

Problem 1

Let $\boldsymbol{a} = (a_n)_{n\geq 0}$ be a sequence of strictly positive real numbers

1. Show that if $\boldsymbol{a}$ is Hankel (resp. Toeplitz) totally positive, then $\boldsymbol{a}$ is log-convex (resp. log-concave).
2. Construct a sequence that is log-convex but not Hankel totally positive.
3. Show that if $\boldsymbol{a}$ is log-concave then $\boldsymbol{a}$ is unimodal.

Problem 2

1. Show that the Catalan numbers satisfy the recurrence equation

$$C_{n+1} = \sum_{i=0}^{n} C_i C_{n-i}, \quad C_0 = 1.$$

2. Deduce that the generating function $C(z) = \sum_{n=0}^{\infty} C_n z^n$ is

$$C(z) = \frac{1 - \sqrt{1-4z}}{2z}.$$

3. Show that the Motzkin numbers M_n satisfy the recurrence equation

$$M_n = M_{n-1} + \sum_{i=0}^{n-2} M_i M_{n-2-i}, \quad M_0 = 1,\ M_1 = 1.$$

4. Deduce that the generating function $M(z) = \sum_{n=0}^{\infty} M_n z^n$ is

$$M(z) = \frac{1 - z - \sqrt{(1+z)(1-3z)}}{2z^2}.$$

Problem 3

We recall that the semi-circular law is the measure supported on $[-2, 2]$ with density

$$f(x) = \frac{1}{2\pi}\sqrt{4 - x^2}\,.$$

1. Using the generating function, show that the n-th Catalan is given by $C_n = \frac{1}{n+1}\binom{2n}{n}$.
2. Deduce that Catalan numbers are log-convex.
3. Show that Catalan numbers are the even moments of the semi-circular law.
4. Deduce that $(C_n)_{n\geq 1}$ is Hankel totally positive.

Problem 4

Let $A = \{a_1, a_2, \ldots, a_n\}$ and $B = \{b_1, b_2, \ldots, b_n\}$ be two sets of vertices in a finite directed weighted acyclic graph. Let $P(a_i, b_j)$ denote the set of all paths from a_i to b_j in the graph, and let $w(p)$ denote the weight of a path p (product of the weights of the edges). Define $F(a_i, b_j) = \sum_{p\in P(a_i,b_j)} w(p)$.

A tuple of paths from A to B is a tuple of paths with starting points a_i and ending points b_j.

Given a tuple of non-intersecting paths that maps a_i to b_j, its weight is the product of the weights of the paths multiplied by the sign of the permutation σ that maps i to $j = \sigma(i)$.

Prove that the sum of the weights of all tuples of non-intersecting paths from A to B is

$$\det\left(F(a_i, b_j)\right)_{1\leq i,j\leq n}\,.$$

Hint:

$$\det\left(F(a_i, b_j)\right)_{1\leq i,j\leq n} = \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma)\prod_{i=1}^{n} F(a_i, b_{\sigma(i)})$$

Problem 5

Let M_n be a matrix from the Gaussian Orthogonal Ensemble (GOE), which consists of $n \times n$ real symmetric matrices whose entries are independent (up to symmetry) and normally distributed with mean 0. Specifically, the entries on the diagonal are distributed as $\mathcal{N}(0, 2)$ and the off-diagonal entries are distributed as $\mathcal{N}(0, 1)$.

The normalized trace of a matrix M is defined as

$$\mathrm{tr}(M) = \frac{1}{n}\sum_{i=1}^{n} M_{ii}.$$

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1. Prove the following

$$\mathbb{E}\left(\mathrm{tr}(M_n^{2k})\right) = n^k C_k(1 + o(1)).$$

2. Deduce an interpretation of the Catalan numbers in terms of large-dimensional limits of GOE matrices.
3. Was the Gaussianity assumption necessary? Come up with a generalization of the result in 2.