Virginia integrable probability summer school 2024

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Problem 1

Let $x_1, x_2, \dots, x_k$ be some k real numbers. Let

$$a_0 = 1,$$

$$a_1 = \sum_i x_i,$$

$$a_2 = \sum_{i_1 < i_2} x_{i_1} x_{i_2},$$

$$a_n = \sum_{1 \le i_1 < i_2 < \dots < i_n \le k} \prod_{j=1}^{n} x_{i_j}.$$

1. Compute the generating function $\sum_{n \ge 0} a_n z^n$.
2. Prove that $(a_n)_{n \ge 0}$ is Toeplitz totally positive if and only if $x_1, \dots, x_k \ge 0$.

Problem 2

Prove that :

$$M(z) = \sum_{n=0}^{\infty} M_n z^n = \cfrac{1}{1 - z - \cfrac{z^2}{1 - z - \cfrac{z^2}{1 - z - \cfrac{z^2}{1 - z - \cfrac{z^2}{1 - \cdots}}}}},$$

$$C(z)=\sum_{n=0}^{\infty} C_n z^n = \cfrac{1}{1-\cfrac{z}{1-\cfrac{z}{1-\cfrac{z}{1-\cfrac{z}{1-\cdots}}}}},$$

and

$$\sum_{n=0}^{\infty} B_n z^n = \cfrac{1}{1-z-\cfrac{z^2}{1-2z-\cfrac{2z^2}{1-3z-\cfrac{3z^2}{1-4z-\cfrac{4z^2}{1-\cdots}}}}},$$

where C_n is the n-th Catalan Number, M_n is the n-th Motzkin number and B_n is the number of set partitions of $[n]$.

Problem 3

- Show that $(2n-1)!!$ counts perfect matchings (fixed points free involutions).
- Deduce that

$$\sum_{n=0}^{\infty} (2n-1)!! z^n = \cfrac{1}{1-\cfrac{z}{1-\cfrac{2z}{1-\cfrac{3z}{1-\cfrac{4z}{1-\cdots}}}}}$$

Problem 4

Prove that any symmetric polynomial can be expressed in a unique way as a polynomial in the elementary symmetric polynomials.

Problem 5

Let $P(x)=\sum_{n\geq 0} a_n x^n$ be a formal power series where a_n are real numbers and $a_0 \neq 0$.

1. Prove that there exists a unique formal power series $\sum_{n\geq 0} b_n x^n$ such that

$$\left(\sum_{n\geq 0} a_n x^n\right)\left(\sum_{n\geq 0} b_n x^n\right) = 1.$$

2. Prove that $b_0 = a_0^{-1}$ and

$$b_k = \frac{(-1)^k}{a_0^{k+1}} \det\left(\begin{bmatrix} a_1 & a_0 & 0 & \cdots & 0 \\ a_2 & a_1 & a_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{k-1} & a_{k-2} & a_{k-2} & \cdots & a_0 \\ a_k & a_{k-1} & a_{k-2} & \cdots & a_1 \end{bmatrix}\right)$$

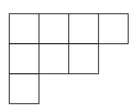
3. Do we need that the (a_n) are real numbers ?

Problem 6

Recall first some definitions

Young Diagrams

A Young diagram of shape $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ is a graphical representation of a partition. It consists of left-aligned rows of boxes, where the i-th row contains λ_i boxes. For example, the Young diagram for $\lambda = (4, 3, 1)$ is:



Semistandard Young Tableaux (SSYT)

A Semistandard Young Tableau (SSYT) of shape λ and entries from $\{1, 2, \ldots, n\}$ is a filling of the Young diagram of λ such that the entries weakly increase across rows and strictly increase down columns.

Schur Polynomials

The Schur polynomial $s_\lambda(x_1, x_2, \ldots, x_n)$ indexed by a partition λ are equivalently defined as:

$$s_\lambda(x_1, x_2, \ldots, x_n) = \sum_T x^T,$$

where the sum is over all SSYT T of shape λ.

1. Prove Jacobi-Trudi formula (using Lindström–Gessel–Viennot lemma).

$$s_\lambda(x_1, \ldots, x_n) = \det\left((h_{\lambda_i+j-i})_{i,j}^{r\times r}\right),$$

   h_i are the complete homogeneous symmetric polynomials.

2. Prove the characterization of Schur positive specializations in terms of totally positive Toeplitz matrices