Virginia integrable probability summer school 2024

Natasha Blitvic Slim Kammoun

June 2024

Problem 1

Let G be a locally finite undirected graph, V_0 be a vertex, and let a_n be the number of walks of length n in G from V_0 to V_0.

1. Suppose first that G is finite. Prove that $(a_n)_{n\geq 0}$ is a moment sequence.
2. Prove that $(a_n)_{n\geq 0}$ is a moment sequence when G is locally finite.
3. Deduce that $\binom{2n}{n}$, Catalan numbers, Motzkin numbers, and $\binom{2n}{n}^2$ are moment sequences.

Problem 2

Consider the Gaussian measure.

$$\frac{\mathrm{d}\mu(x)}{\mathrm{d}x} = \frac{\mathrm{e}^{-x^2/2}}{\sqrt{2\pi}}.$$

and recall that Hermite polynomials are defined by

$$H_n(x) = (-1)^n e^{\frac{x^2}{2}} \frac{d^n}{dx^n} e^{-\frac{x^2}{2}}$$

1. Prove the two-term recurrence equation for the Hermite polynomials:
$$H_{n+1}(x) = xH_n(x) - nH_{n-1}(x), \quad H_0(x) = 1, \quad H_1(x) = x.$$
2. Prove that $(H_n)_{n\geq 1}$ is the family of orthogonal polynomials associated the Gaussian measure.
3. Compute the associated moments:
$$\mu_n = \int_{-\infty}^{\infty} x^{2n} \frac{\mathrm{e}^{-x^2/2}}{\sqrt{2\pi}}\, \mathrm{d}x.$$

Problem 3

Suppose that x and y are freely independent.

1. Prove that : $\varphi(xy) = \varphi(x)\varphi(y)$
2. Compute $\varphi(xyx^2y)$.

Problem 4

Prove that if $(P_n)_{n\geq}$ are the orthogonal polynomials w.r.t to a linear functional then :

$$P_{n+1}(x) = (x + B_n)P_n(x) - C_nP_{n-1}(x), \quad n \geq 1.$$

Problem 5

1. Prove the formula seen in the lecture: When decomposing into set partitions :

   $$A(x) = \exp(I(x))$$

   Where

   $$A(x) = \sum \frac{a_n x^n}{n!} \text{ "all set partitions"}$$
   $$I(x) = \sum \frac{i_n x^n}{n!} \text{ "irreducibles"}$$

2. Deduce that if a_n are the (classical) moments of a random variable than i_n are the (classical) cumulants of the same random variable.

Problem 6

Consider the Hilbert space $\ell_2 = \ell_2(\mathbb{N}_0)$ formed of complex sequences $a = (a_n)_{n\geq 0}$ satisfying $\sum_{n\geq 0} |a_n|^2 < \infty$, endowed with the inner product $\langle a, b\rangle = \sum_{n\geq 0} a_n \overline{b_n}$. It is easy to check that the elements $(e_n)_{n\geq 0}$,

$$e_n := (\delta_{1,n}, \delta_{2,n}, \delta_{3,n}, \ldots) = (0, \ldots, 0, 1, 0, \ldots)$$

form an orthonormal basis of ℓ_2.

Consider the operator R defined on the basis elements as follows:

$$Re_n = e_{n+1}.$$

It is easy to check that R extends to a bounded operator on all of ℓ^2. Furthermore, it is easy to check that its adjoint R^* is determined by $Re_0 = 0$ and $R^*e_n = e_{n-1}$ for $n \geq 1$.

Consider the linear functional $E : \mathcal{B}(\ell^2) \to \mathbb{C}$, given by $E(S) = \langle Se_0, e_0\rangle$. It is easy to check that this is a positive linear functional.

1. (If you're familiar with Hilbert spaces, prove the aforementioned "easy to check" assertions.

2. Let $S = R + R^*$. Compute the moments $E(S^n)$. What is the law of S?

3. Connect this result to Problem 5 of Problem Set 1. What can you conclude?