# Virginia integrable probability summer school 2024

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## Problem 1

We denote by $a_n^\pi$ the number of permutations of size $n$ that avoid the classical pattern $\pi$.

1. Prove that the number of permutations that avoid the classical pattern 132 satisfies the relation

$$a_{n+1}^{132} = \sum_{i=0}^{n} a_i^{132} a_{n-i}^{132}, \quad a_0^{132} = 1.$$

2. Deduce that they are moment sequences

3. (hard) Prove that for any patterns $\pi$ of length 3, $(a_n^\pi)_n$ is a moment sequence.

4. Check that the numbers of permutations that avoid the consecutive pattern 132 are not moment sequences.

## Problem 2

We recall that a descent is a position $i$ such that $\sigma(i+1) < \sigma(i)$. Let $E(n,k)$ be the number of permutations of size $n$ with $k$ descents, also known as Eulerian numbers. Prove that for any $t > 0$,

$e_n = \sum_{k=0}^{n} E(n,k)\, t^k$ as a sequence on $n$ is a moment sequence of a known distribution.

Hint: First prove that

$$E(n,k) = (n-k)E(n-1,k-1) + (k+1)E(n-1,k).$$

## Problem 3

A $k$-arrangement is a permutation where fixed points are colored with a color in $\{1, 2, \ldots, k\}$. For example 0-arrangements are derangements and 1-arrangement are permutations.

1. Prove that the number of $k-$arrangements of size $n$ as a sequence on $n$ is a moment sequence and describe the corresponding measure.