Acrtic Curves of the T-system with Slanted Initial Data

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Definition and Visualization of T-system

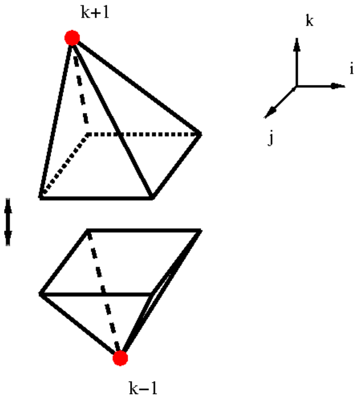
Definition

The **T-system** of type A_∞, also known as **octahedron recurrence**, is a 3-dimensional recurrence relation in variables $T_{i,j,k}$, $i, j, k \in \mathbb{Z}$:

$$T_{i,j,k+1}\, T_{i,j,k-1} = T_{i+1,j,k}\, T_{i-1,j,k} + T_{i,j+1,k}\, T_{i,j-1,k}.$$

Definition and Visualization of T-system

$$T_{i,j,k+1}\, T_{i,j,k-1} = T_{i+1,j,k}\, T_{i-1,j,k} + T_{i,j+1,k}\, T_{i,j-1,k}.$$



Stepped Surface

$$T_{i,j,k+1}\, T_{i,j,k-1} = T_{i+1,j,k}\, T_{i-1,j,k} + T_{i,j+1,k}\, T_{i,j-1,k}.$$

The **solution** $T_{i,j,k}$ (an expression as function of initial data) is unique once we fix admissible initial data along any given *"stepped surface"* $\mathbf{k} := \{(i_0, j_0, k_{i_0,j_0}) : i_0, j_0 \in \mathbb{Z}, |k_{i+1,j} - k_{i,j}| = |k_{i,j+1} - k_{i,j}| = 1\}$ for all $i, j \in \mathbb{Z}$. **The initial data assignments** read

$$T_{i_0,j_0,k_{i_0,j_0}} = t_{i_0,j_0}\,, \qquad (i_0, j_0 \in \mathbb{Z}) \tag{2.1}$$

for some fixed initial variables t_{i_0,j_0}, $i_0, j_0 \in Z$.

Stepped surface as tessellation

For a stepped surface **k** of initial data, we first bicolor the faces such that any 2 neighboring faces of the half-octahedron have different color. We further record the k-coordinates of the points in **k**, based on the dictionary

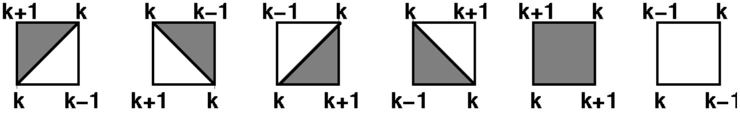
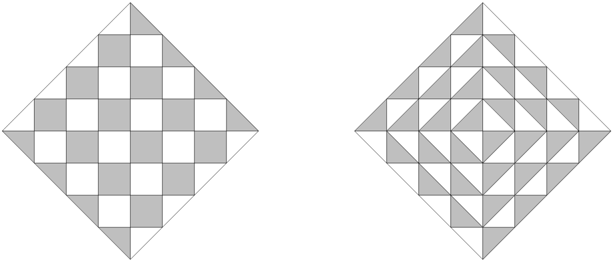


Figure: Tessellation rule

Example



(a) (b)

Figure: (a) is the domain for "flat" stepped surface where $k = 0$ or 1 while (b) is a stepped surface in the shape of a pyramid in 3-dimension with apex at height $k = 5$

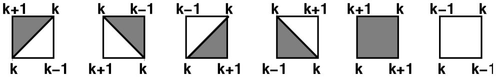


Figure: Tessellation rule

From tessellation to $T_{i,j,k}$

To find solution $T_{i,j,k}$, we consider the intersection between the cone $|x-i|+|y-j| \leq |z-k|$, $x, y, z \in \mathbb{Z}_{>0}$ with apex (i,j,k) and the initial data stepped surface $\mathbf{k}$.
$T_{i,j,k}$ expression contains only the initial data point inside this intersection

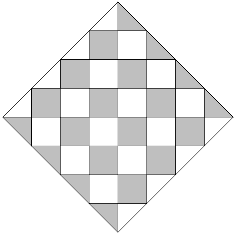
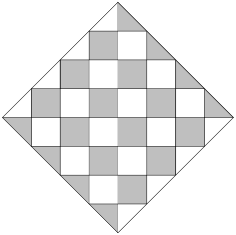
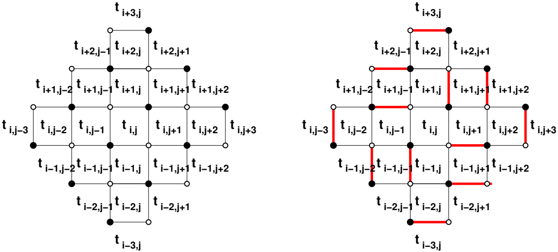


Figure: The intersection of the cone with apex $(0,0,4)$ in the solution of $T_{0,0,4}$

Dimer formulation





(a) **(b)**

Slanted Initial Data

Definition (Slanted initial data)

Given a triple $(r, s, t) \in \mathbb{Z}^3_{\geq 0}$, $\gcd(r, s, t) = 1$ *and* $t > \max(r, s)$, *we consider the initial data lies on* $2t$ *consecutive planes* (P_m), $m = 0, 1, 2, \dots, 2t - 1$.

$$(P_m) = \{(i, j, k) \mid ri + sj + tk = m\}$$

Slanted initial data visualization

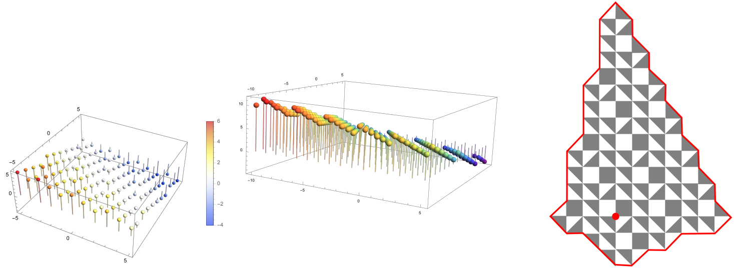


Figure: Stepped surface of initial data on the parallel planes perpendicular to $(r, s, t) = (1, 2, 3)$ and the tessellation corresponding to the solution $T_{0,0,6}$

The solution of T-system with slanted initial data

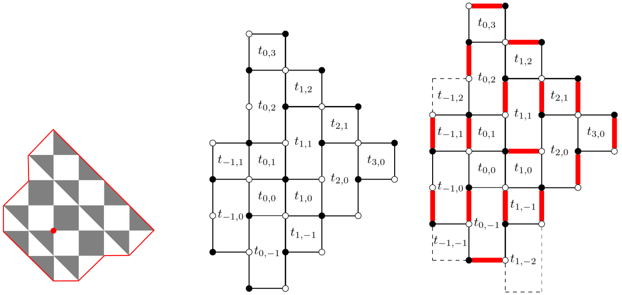
Theorem (Di Francesco, V. 24')

The solution of the T-system with slanted initial data is expressed as:

$$T_{i,j,k} = \sum_{\substack{\text{dimer configs. } D \\ \text{on } \mathcal{G}}} \prod_{\substack{\text{faces } (x,y) \\ \text{of } G}} \begin{cases} (t_{x,y})^{v_{x,y}/2-1-N_{x,y}(D)} & (x,y) \text{ interior faces} \\ (t_{x,y})^{1-N_{x,y}(D)} & (x,y) \text{ boundary faces} \end{cases}$$

where the sum extends over all dimer configurations on the dual graph $\mathcal{G}$, (a class of planar lattice graph defined by Bousquet-Melou, Propp, West called "pinecone" in the original literature), while $v_{x,y} \in \{4,6\}$ is the valency of the face (x,y) and $N_{x,y}(\mathcal{D}) \in \{0,1,...,v_{x,y}/2\}$ denotes the number of dimers occupying edges adjacent to the face (x,y).

Example

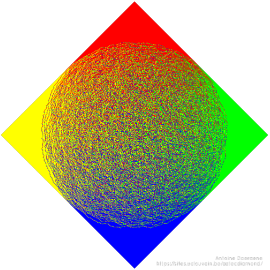


(a) (b) (c)

Figure: The tessellated domain (A) for $T_{0,0,4}$ with $(1,1,3)$-slanted initial data, the corresponding dual graph (B) $\mathcal{G}^{0,0,4}_{1,1,3}$ and a sample dimer configuration (C) corresponds to the contribution $\dfrac{t_{-1,-1}t_{-1,2}t_{0,0}t_{1,-2}t_{2,0}t_{2,2}}{t_{-1,1}t_{0,-1}t_{1,-1}t_{1,1}t_{2,1}}$ to the partition function $T_{0,0,4}$

Density of T-system

- The solution $T_{i,j,k}$ of the T-system is the **partition function** of some suitable dimer model $\mathcal{D}$ in terms of the initial data.
- Using some data from the solution $T_{i,j,k}$, one can study the **the arctic phenomenon**, which is the behavior of the model in large size limit, where one can find a separation between crystal phases and liquid phase in the dimer configuration



Density of T-system

Pick a point $(i_0, j_0, k_0 = k_{i_0,j_0})$ belonging to one of the initial data planes (P_m). Then (i_0, j_0, k_0) corresponds to the center of a $2v$-valent face at coordinates (i_0, j_0) in the dual dimer graph centered at (i, j). As the local contribution to the partition function is $(t_{i_0,j_0})^{\frac{v_{i_0,j_0}}{2}-1-N_{i_0,j_0}(\mathcal{D})}$, we may write:

$$\begin{aligned}
\rho_{i,j,k}^{(i_0,j_0,k_0)} &:= t_{i_0,j_0}\partial_{t_{i_0,j_0}} \log(T_{i,j,k})\Big|_{t_{i_0,j_0}} \\
&:= \frac{1}{T_{i,j,k}}\, t_{i_0,j_0}\partial_{t_{i_0,j_0}}(T_{i,j,k}) \\
&= \langle \frac{v_{i_0,j_0}}{2} - 1 - N_{i_0,j_0}(\mathcal{D})\rangle_{i,j,k}
\end{aligned}$$

where $\langle f\rangle_{i,j,k}$ stands for the statistical average of the function f over the dimer configurations on $\mathcal{D}$.

When considering the domains of large size, $\rho = 0$ in the crystal/frozen regions.

Frozen regions example

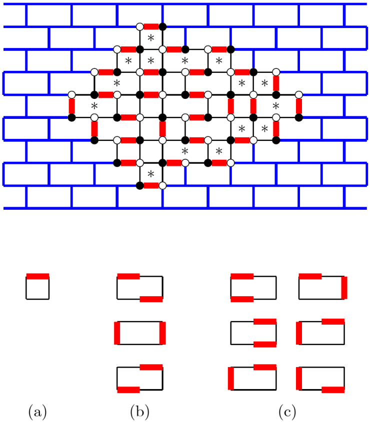


Figure: The four frozen dimer configurations (up to rotation) in the four corners of the $(r, s, t) = (1, 0, 3)$-dual graph. In a larger case, one should see that these frozen facets concentrate at the 4 corners of the dual graph.

Recurrence for ρ

From the octahedron recurrence

$$T_{i,j,k+1}\,T_{i,j,k-1}=T_{i+1,j,k}\,T_{i-1,j,k}+T_{i,j+1,k}\,T_{i,j-1,k}.$$

The recurrence relation for ρ in general is:

$$\rho^{(i_0,j_0,k_0)}_{i,j,k+1}+\rho^{(i_0,j_0,k_0)}_{i,j,k-1}=L_{i,j,k}\,(\rho^{(i_0,j_0,k_0)}_{i+1,j,k}+\rho^{(i_0,j_0,k_0)}_{i-1,j,k})+R_{i,j,k}\,(\rho^{(i_0,j_0,k_0)}_{i,j+1,k}+\rho^{(i_0,j_0,k_0)}_{i,j-1,k}),$$

where

$$L_{i,j,k}:=\frac{T_{i+1,j,k}\,T_{i-1,j,k}}{T_{i,j,k+1}\,T_{i,j,k-1}}.$$

$$R_{i,j,k}=\frac{T_{i,j+1,k}\,T_{i,j-1,k}}{T_{i,j,k+1}\,T_{i,j,k-1}}=1-L_{i,j,k}$$

"Uniform" Initial Data

Let $\alpha > 1$ is the solution of $\alpha^{t^2} = \alpha^{r^2} + \alpha^{s^2}$. Define **uniform** initial data to be $t_{i_0,j_0} = \alpha^{m(m-1)/2}$ where m encoded the parallel planes m where $(i_0, j_0) \in P_m$, $m \in 0, \ldots 2t-1$. Then,

$$T_{i,j,k} = \alpha^{m(m-1)/2}, m = ri + sj + tk$$

One can compute:

$$L_{i,j,k} = \alpha^{r^2-t^2}, \qquad R_{i,j,k} = \alpha^{s^2-t^2},$$

$$\begin{aligned}\rho_{i,j,k+1} + \rho_{i,j,k-1} &= \alpha^{r^2-t^2}(\rho_{i+1,j,k} + \rho_{i-1,j,k}) \\ &\quad + \alpha^{s^2-t^2}(\rho_{i,j+1,k} + \rho_{i,j-1,k}).\end{aligned}$$

Generating Function

Putting these values $\rho_{i,j,k}^{(i_0,j_0,k_0)}$ into a generating function:

$$\rho^{(i_0,j_0,k_0)}(x,y,z) := \sum_{i,j,k\in\mathbb{Z}} \rho_{i,j,k}^{(i_0,j_0,k_0)}\, x^i\, y^j\, z^k.$$

Using the linear recurrence relation for ρ and the initial conditions $\rho_{i,j,k}^{(i_0,j_0,k_0)} = \delta_{i,i_0}\delta_{j,j_0}\delta_{k,k_0}$ along the initial data surface $\mathbf{k}$ gives $\rho^{(i_0,j_0,k_0)}(x,y,z)$:

$$\begin{aligned}1-\rho^{(0,0,0)}(x,y,z) &= \frac{z^2}{1+z^2-z\alpha^{r^2-t^2}\left(x+\frac{1}{x}\right)-z\alpha^{s^2-t^2}\left(y+\frac{1}{y}\right)}\\ &:= \frac{z^2}{D_{r,s,t}(x,y,z)}\end{aligned}$$

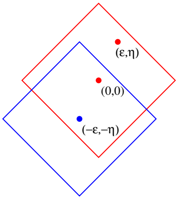
But this is not really what we need, at least not yet

Translational invariance

The quantity $\rho_{i,j,k}^{(i_0,j_0,k_0)}$ only measures the dimer density of the face (i_0,j_0) of the dual graph centered at (i,j) with size k.

But this is enough by the symmetry of the flat initial data

$$\rho_{i,j,k}^{(\epsilon,\eta)} = \rho_{0,0,k}^{(\epsilon-i,\eta-j)}$$



Translational Invariance to Generating Function

Without going into detail, the translational invariance results in a change of variable in the generating function. Specifically, the generating function for the density $\rho_{i,j,k}^{(i_0,j_0,k_0)}$ **over all** (i_0, j_0, k_0) is equivalent to the generating function

$$\rho^{(i_0,j_0,k_0)}(x,y,z) := \sum_{\substack{i,j,k\in\mathbb{Z} \\ \mu(i,j,k)\geq 0}} \rho_{i,j,k}^{(i_0,j_0,k_0)}\, x^i\, y^j\, z^k.$$

with a change of variable $x \mapsto z^{r/t}x^{-1}$, $y \mapsto z^{s/t}y^{-1}$ and some factors.

Asymptotic of $\rho(x, y, z)$ via ACSV

The asymptotic of the density is governed by denominator of its generating function, which vanishes as x, y, z approach 1

$$\Delta_{r,s,t}(x, y, z) := D_{r,s,t}(z^{r/t}x^{-1}, z^{s/t}y^{-1}, z)$$

Consider the **scaling limit** when $i, j, k \to \infty$ with $i/k \to u$, $j/k \to v$ finite, we now apply the method of multivariate generating functions by Baryshnikov-Pemantle-Wilson [BP11, PW04, PW08] for conical singularities, letting $x \mapsto e^{\epsilon x}$, $y \mapsto e^{\epsilon y}$ and $z \mapsto e^{-\epsilon(ux+vy)}$ and expanding at leading order in ϵ, we find

$$\Delta_{r,s,t}(e^{\epsilon x}, e^{\epsilon y}, e^{-\epsilon(ux+vy)}) = \epsilon^2 H_{r,s,t}(x, y, u, v) + O(\epsilon^4)$$

for some explicit polynomial H of x, y, u, v

The Use of ACSV

Further imposing $H_{r,s,t}(x,y) = \partial_x H_{r,s,t}(x,y) = \partial_y H_{r,s,t}(x,y) = 0$ has non-trivial solution in x, y, which leads to the vanishing condition of the Hessian determinant:

$$P(u,v) = det(\begin{pmatrix} \partial_x^2 H & \partial_x \partial_y H \\ \partial_y \partial_x H & \partial_y^2 H \end{pmatrix}) = 0$$

This $P(u,v)$ is the curve that separates the phase where in the limit $\rho_{i,j,k}^{i_0,j_0,k_0} = 0$ and liquid phase where $\rho_{i,j,k}^{i_0,j_0,k_0} \neq 0$

Arctic curves for "uniform" slanted initial data

Theorem (Di Francesco, V. 24')

The arctic curves of typical large size (r,s,t)-slanted dual graph associated to the solution of the T-system with uniform initial data $t_{i,j} = \alpha^{m(m-1)/2}$ on each slanted plane $m = ri + sj + tk = 0, 1, \cdots, 2t-1$, is the ellipse

$$(1-A)\, t^2 u^2 + A\, t^2 v^2 - A(1-A)\,(ru + sv + t)^2 = 0$$

where $A = A_{r,s,t} := \alpha^{r^2 - t^2}, 1 - A = \alpha^{s^2 - t^2}$ inscribed in the scaling domain

$$v = -\frac{t}{t+s} - \frac{t+r}{t+s}u, \qquad v = -\frac{t}{s-t} - \frac{t+r}{s-t}u \qquad \left(u \in \left[-\frac{t}{t+r}, 0\right]\right)$$

$$v = -\frac{t}{t+s} + \frac{t-r}{t+s}u, \qquad v = -\frac{t}{s-t} + \frac{t-r}{s-t}u \qquad \left(u \in \left[0, \frac{t}{t-r}\right]\right).$$

Example

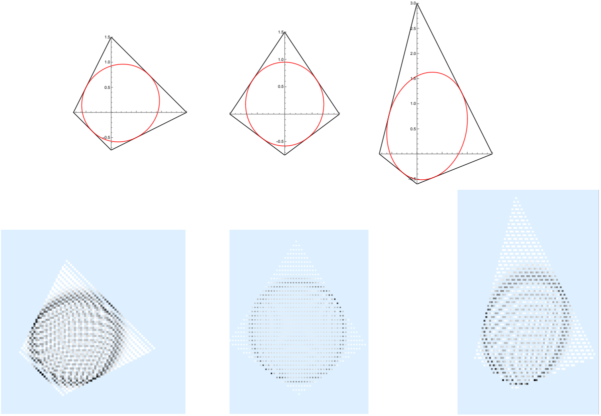


Figure: Arctic curves for r, s, t-dual graph, together with the scaled domain. Left: $(r, s, t) = (1, 1, 3)$, center: $(r, s, t) = (0, 1, 3)$, right $(r, s, t) = (1, 2, 3)$.

"Periodic" Slanted Initial Data

A specific initial data on the stack of slanted $(r, s, t) = (r, r, t)$-planes, which has periodicity two in both i and j direction, namely, on each slanted plane P_m, $m \in \{0, 1, \cdots, 2t-1\}$:

$$t_{i_0, j_0} = \alpha^{m(m-1)/2} \times \begin{cases} a & (i_0 = 0, j_0 = 0 \;\; mod \;\; 2) \\ b & (i_0 = 0, j_0 = 1 \;\; mod \;\; 2) \\ c & (i_0 = 1, j_0 = 0 \;\; mod \;\; 2) \\ d & (i_0 = 1, j_0 = 1 \;\; mod \;\; 2) \end{cases}$$

such that $ri_0 + sj_0 + tk_0 = m$, $i_0 + j_0 + k_0 = 0 \;\; mod \;\; 2$, and with $\alpha = 2^{\frac{1}{t^2 - r^2}}$ as in previous "uniform" initial data.

Question: Can we also analyze the asymptotic via ACSV?

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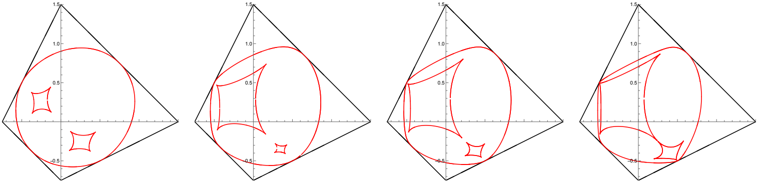
Question: Can we also analyze the asymptotic via ACSV?

Quick Answer: Yes, but it is not easy and only $r = s$ case has been solved

Arctic Curves Profile

Let $\tau = \frac{a^2}{a^2+d^2}$ and $\sigma = \frac{b^2}{b^2+c^2}$ where a, b, c, d are the periodic weights on each plane. Then the quantities $\mathcal{L}_{i,j,k}$ and $\mathcal{R}_{i,j,k}$ only depends on τ and σ, thus also controling the artic curves:



(a) $\sigma = 1/4, \tau = 1/2$

(b) $\sigma = 1/4, \tau = 1/8$

(c) $\sigma = 1/4, \tau = 1/16$

(d) $\sigma = 1/4, \tau = 1/64$

Figure: Arctic curves for $(r, s, t) = (1, 1, 3)$, together with the scaled domain with fixed $\sigma = 1/4$.

Arctic Curves Profile

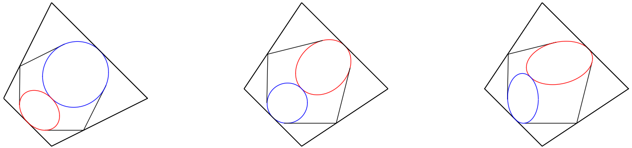
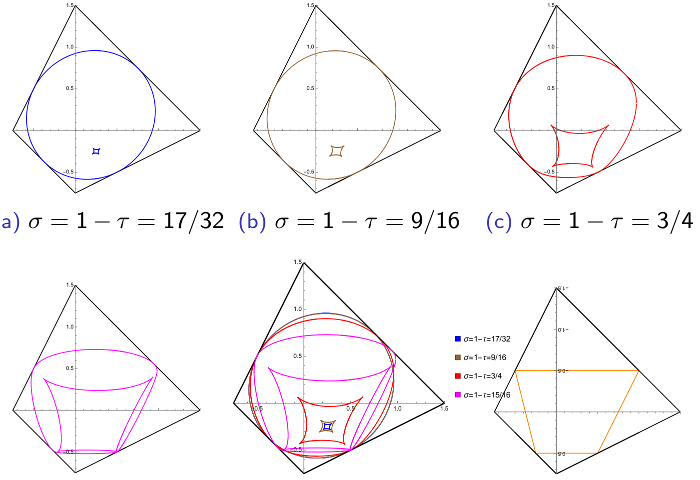


Figure: Arctic curves for r, s, t-dual graph, together with the scaled domain for fixed $\tau = 0$. Left: $(r, s, t) = (1, 1, 3), \sigma = \frac{1}{2}$, Center: $(r, s, t) = (1, 1, 5), \sigma = \frac{1}{2}$, Right: $(r, s, t) = (1, 1, 5), \sigma = \frac{2}{3}$.

Arctic Curves Profile



(a) $\sigma = 1 - \tau = 17/32$ (b) $\sigma = 1 - \tau = 9/16$ (c) $\sigma = 1 - \tau = 3/4$

(d) $\sigma = 1 - \tau = 15/16$ (e) Curves (A-D) (f) $\sigma = 1 - \tau = 0$

Figure: Arctic Curves for $(r, s, t) = (1, 1, 3)$ and $\sigma = 1 - \tau$

Holographic Principle

What do we have now:

- New exact solutions of the T-system, with (r, s, t)-initial data specified along parallel planes prependicular to some direction (r, s, t).
- Some technical framework that said if one can find the general explicit solution for the coefficient $\mathcal{L}$ and $\mathcal{R}$ in the density recurrence relation $\rho_{i,j,k}$ then the arctic curves can be obtained via the sigularities.

A question: Starting from a solution of the T-system for some (r, s, t)-initial data value, can we re-interpret the *same* solution as corresponding to some different stepped surface along some $(\tilde{r}, \tilde{s}, \tilde{t})$-planes? Can the arctic curves even be obtained?

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Answer: Yes, definitely

Holographic Example

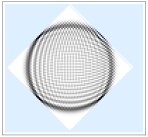
Consider the simplest case of uniform initial data viewed from the "flat" perspective with $(\tilde{r}, \tilde{s}, \tilde{t}) = (0, 0, 1)$ - the dual graph will the Aztec diamond. But now, in the projection onto (i, j), the weight reads:

$$T_{i,j,0} = \alpha^{\frac{(ri+sj)(ri+sj-1)}{2}} \qquad (i + j = 0 \bmod 2)$$
$$T_{i,j,1} = \alpha^{\frac{(ri+sj+t)(ri+sj+t-1)}{2}} \qquad (i + j = 1 \bmod 2)$$

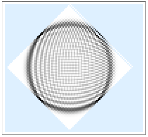
This is equivalent to the solution of the T-system interpreted as partition function for dimers on $(\tilde{r}, \tilde{s}, \tilde{t})$ graph (Aztec Diamond in this case), whose limit shape is governed by the *same* equations as the original (r, s, t) setting, i.e. the asymptotics are dictated by

$$\Delta^{\tilde{r},\tilde{s},\tilde{t}}_{r,s,t}(x, y, z) = D_{r,s,t}(z^{\frac{\tilde{r}}{\tilde{t}}}/x, z^{\frac{\tilde{s}}{\tilde{t}}}/y, z)$$

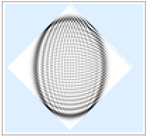
Flat Holographic



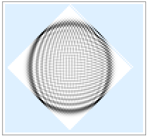
(a)
$(r, s, t) = (1, 1, 3)$



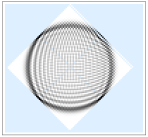
(b)
$(r, s, t) = (1, 2, 5)$



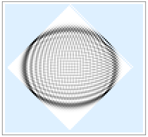
(c)
$(r, s, t) = (1, 7, 9)$



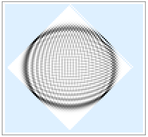
(d)
$(r, s, t) = (1, 7, 15)$



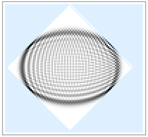
(e)
$(r, s, t) = (2, 2, 9)$



(f)
$(r, s, t) = (5, 0, 8)$



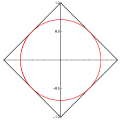
(g)
$(r, s, t) = (7, 0, 13)$



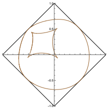
(h)
$(r, s, t) = (3, 0, 4)$

Figure: Density profile for given (r, s, t)-slanted values viewed from the $(0, 0, 1)$ perspective on the scaled domain $|u| + |v| = 1$. We have represented the coefficient $\rho_{i,j,k}$ of the generating series $\rho(x, y, z)$ of order $k = 45$ and $k = 46$, written as a function of $(i/k, j/k)$, for $i \in \{-45, \cdots, 45\}$ and $j \in \{-45, \cdots, 52\}$

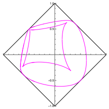
More Examples



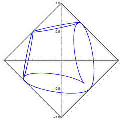
(a) $\tau = \sigma = 1/2$



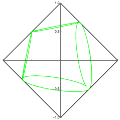
(b) $\tau = \sigma = 1/4$



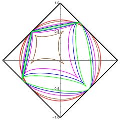
(c) $\tau = \sigma = 1/10$



(d) $\tau = \sigma = 1/20$



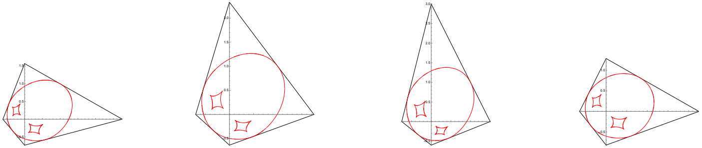
(e) $\tau = \sigma = 1/40$



(f) Curves (A-E)

Figure: Arctic curves library for 2×2 periodic weights on $(r, s, t) = (1, 1, 3)$, $\tau = \sigma$ varied from the $(\tilde{r}, \tilde{s}, \tilde{t}) = (0, 0, 1)$-Aztec Diamond point of view

More Holographic Examples



(a) $(\tilde{r}, \tilde{s}, \tilde{t}) = (7, 4, 11)$

(b) $(\tilde{r}, \tilde{s}, \tilde{t}) = (3, 4, 7)$

(c) $(\tilde{r}, \tilde{s}, \tilde{t}) = (1, 2, 3)$

(d) $(\tilde{r}, \tilde{s}, \tilde{t}) = (5, 2, 9)$

Figure: Several $(\tilde{r}, \tilde{s}, \tilde{t})$ views of arctic curves of the $(1, 1, 3)$ slanted 2x2 periodic solution with $\tau = 1/4$, $\sigma = 1/2$.

Holographic Principle Interpretation

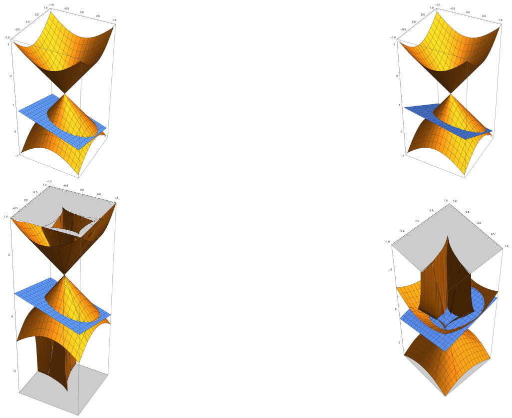
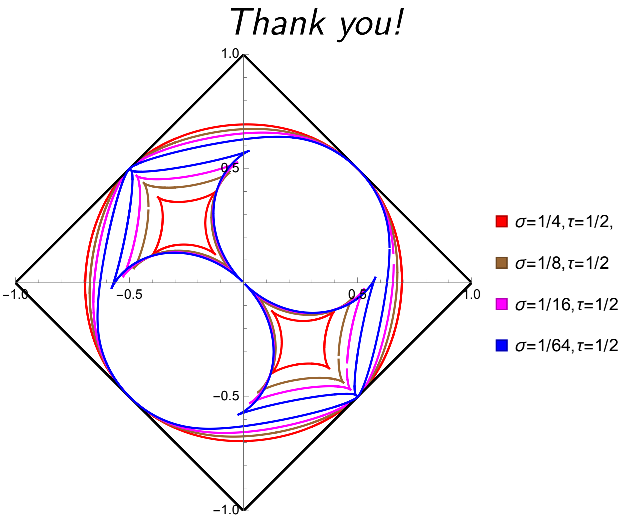


Figure: Top: The surface for the T-system with (1,1,3)-slanted uniform initial data (depicted in orange/brown), together with Left: the original slanted initial data plane $\mathcal{P}_{1,1,3}$ (depicted in blue); Right: the holographic section in the direction $(1, 3, 5)$, i.e. the plane $\mathcal{P}_{1,3,5}$ (depicted in blue). Bottom: The surface for the T-system with (1,1,3)-slanted 2x2-periodic initial data with $\sigma = \tau = 1/4$ (depicted in orange/brown), and the original slanted initial data plane $\mathcal{P}_{1,1,3}$ (depicted in blue). Left: upper side view. Right: Lower side view.

Thank you


Thank you!
1.0
0.5
−1.0
−0.5
0.5
1.0
−0.5
−1.0
σ=1/4, τ=1/2,
σ=1/8, τ=1/2
σ=1/16, τ=1/2
σ=1/64, τ=1/2

References I

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[PW04] Robin Pemantle and Mark C. Wilson.
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